Scale dependence and renormalon-inspired resummations for some QCD observables

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Abstract

Since the advent of Quantum Field Theory (QFT) in the late 1940's, perturbation theory has become one of the most successful means of extracting phenomenologically useful information from QFT. In the ever-increasing enthusiasm for new phenomenological predictions, the mechanics of perturbation theory itself have taken a back seat. It is in this light that this thesis aims to investigate some of the more fundamental properties of perturbation theory.

In the first part of this thesis, we develop the idea, suggested by C.J.Maxwell, that at any given order of Feynman diagram calculation for a QCD observable *all* renormalization group (RG)-predictable terms should be resummed to all-orders. This "complete" RG-improvement (CORGI) serves to separate the perturbation series into infinite subsets of terms which when summed are renormalization scheme (RS)-invariant. Crucially all ultraviolet logarithms involving the dimensionful parameter, Q, on which the observable depends are resummed, thereby building the correct Q-dependence. We extend this idea, and show for moments of leptoproduction structure functions that all dependence on the renormalization and factorization scales disappears provided that all the ultraviolet logarithms involving the physical energy scale Q are completely resummed. The approach is closely related to Grunberg's method of Effective Charges.

In the second part, we perform an all-orders resummation of the QCD Adler D-function for the vector correlator, in which the portion of perturbative coefficients containing the leading power of b, the first beta-function coefficient, is resummed to all-orders. To avoid a renormalization scale dependence when we match the resummation

to the exactly known next-to-leading order (NLO), and next-NLO (NNLO) results, we employ the Complete Renormalization Group Improvement (CORGI) approach , removing all dependence on the renormalization scale. We can also obtain fixed-order CORGI results. Including suitable weight-functions we can numerically integrate these results for the D-function in the complex energy plane to obtain so-called "contour-improved" results for the ratio R and its tau decay analogue R_τ. We use the difference between the all-orders and fixed-order (NNLO) results to estimate the uncertainty in $\alpha_s(M_Z^2)$ extracted from R_τ measurements, and find $\alpha_s(M_Z^2) = 0.120 \pm 0.002$. We also estimate the corresponding uncertainty in $\alpha(M_Z^2)$ arising from hadronic corrections by considering the uncertainty in $R(s)$, in the low-energy region, and compare with other estimates. Analogous resummations are also given for the scalar correlator. As an adjunct to these studies we show how fixed-order contour-improved results can be obtained analytically in closed form at the two-loop level in terms of the Lambert W-function and hypergeometric functions.

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Declaration

I declare that no material presented in this thesis has previously been submited for a degree at this or any other university.

The research presented in this thesis has been carried out with Dr. C.J.Maxwell. Aspects of Chapters 3 and 5 are based on the following publications:

- Complete Renormalization Group Improvement- Avoiding Factorization and Renormalization Scale Dependence in QCD Predictions.
C.J.Maxwell and A.Mirjalili, Nucl.Phys. **B577** (2000) 209.

- Renormalon-inspired resummations for vector and scalar correlators- estimating the uncertainty in $\alpha_s(m_\tau^2)$ and and $\alpha(M_Z^2)$.
C.J.Maxwell and A.Mirjalili, Nucl.Phys. **B611** (2001) 423.

To my Dad and my Family

Contents

Chapter 1

Framework of QCD

1.1 Strong Interaction Theory

The strong interaction of fundamental particles has been successfully described by a non-abelian gauge field theory called Quantum Chromodynamics (QCD). Because of its outstanding property of asymptotic freedom, a perturbative treatment makes sense for short distance phenomena. At short distances, equivalently high energies, the effective coupling is small and the theory can be studied using perturbative techniques. In dealing with high-order effects one immediately [1] finds that the calculated results crucially depend on the way one renormalizes the divergent integrals appearing in the calculation. This is called the renormalization-scheme dependence of perturbative predictions.

In addition to renormalizability, the gauge principle is a powerful technique to reveal symmetry features of the theory which consequently will interrelate gauge fields (photons and gluons). QCD together with the Electroweak theory comprise the SU(3)×SU(2) ×U(1) Standard Model of elementary particles. The Standard Model asserts that the material in the universe is made up of elementary fermions interacting through fields, of which they are the sources. The particles associated with the interaction fields are bosons. The elementary fermions of the standard model are of two types: leptons and quarks. All have spin $\frac{1}{2}$, in units of $\hbar$, and in isolation would be described by the Dirac

equation. Leptons interact only through the electromagnetic interactions and the weak interactions. Quarks interact through the electromagnetic and weak interactions and also through the strong interaction. The four types of interaction fields are set out in Table 1.

When we consider the Standard Model, the mass terms of fermion fields cause some complications. They will break the global gauge invariance of the Standard Model due to the left and right-handed components that have different gauge quantum numbers, and at this stage the Higgs field will appear as a consequence of spontaneous symmetry breaking where its existence is essential as well for renormalizability of the theory. In the following we treat QCD separately and insert a Dirac mass term, however this will turn out to be surplus to our needs, and typically we shall take the approximation of massless quarks.

We shall begin by giving a brief introduction to QCD, considering the Lagrangian and the origin of each term and the Feynman rules. A detailed discussion of QCD theory and phenomenology can be found in reference [2–4]. A historical background to the development of QCD can be found in reference [5].

Table 1: Types of interaction filelds

Interaction field	*Boson*	*Spin*
Gravitational field	*Gravitons*	2
Weak field	W^+, W^-, Z *particles*	1
Electromagnetic field	*Photons*	1
Strong field	*Gluons*	1

1.2 QCD as a non-abelian Gauge theory

Gauge field theories are a particular kind of field theory based on the gauge principle. The gauge principle is the requirement that the theory be invariant under local gauge transformations.

As a simple application of this principle in Electrodynamics, the electromagnetic field appears as a consequence of the invariance of the Lagrangian under local gauge symmetry transformations. The electric charges Q obey the commutative (abelian) algebra corresponding to the U(1) group. In extending the algebra to a more general non-commutative (non-Abelian) one, Yang-Mills theory is obtained. In this theory quarks appear as fundamental particles in QCD and gluons will be the gauge mediating particles.

The Pauli exclusion principle requires that quarks have an additional degree of freedom that we call colour. There are three different possible colours (charges) and for practical purposes these are chosen to be primary colours (red, green and blue). Therefore the correct symmetry group for the strong interactions must be SU(3). The quarks are represented by vectors in the three-dimensional colour space and transform under the fundamental representation of the group. Since the quarks are fermions, each of these vector components will be a Dirac spinor $\Psi(x)$ in Lorentz space. A transformation in colour space is then given by

$$\Psi_a \longrightarrow \Psi'_a = U_a^b(x)\Psi_b, \tag{1.1}$$

where U_a^b is a group element of SU(3). It is straightforward to extend the analysis to a general numbers of colours, N_c, by considering the group SU(N_c). The transformation martices U are $N_c \times N_c$ unitary matrices ($UU^\dagger$=1) with |U|=1 which can be written as

$$U(x) = e^{iT^a\theta^a(x)}, \tag{1.2}$$

where the T^a, $a = 1, 2, ..., N_c^2 - 1$, are generators of the Lie algebra and satisfy the comutation relation

$$[T^a, T^b] = if^{abc}T^c, \tag{1.3}$$

which defines the real and antisymmetric constants f^{abc}, the structure constants of the group. The local gauge principle demands that the theory is invariant under the gauge transformation of Eq.(1.1).

1.3 Lagrangian of QCD

We begin with a brief description of the QCD Lagrangian and the Feynman rules which can be derived from it. Just as in Quantum Electrodynamics (QED) the perturbative calculation of any process requires the use of Feynman rules describing the interactions of quarks and gluons. The Feynman rules required for a perturbative analysis of QCD, can be derived from the Lagrangian density which is given by

$$\mathcal{L}_{QCD} = \mathcal{L}_{classical} + \mathcal{L}_{gauge-fixing} + \mathcal{L}_{ghost}. \tag{1.4}$$

The expression for the classical Lagrangian density is

$$\mathcal{L}_{classical} = \frac{-1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \sum_{k=1}^{N_f} \overline{\psi}_k (i\gamma^\mu D_\mu - m_k)\psi_k, \tag{1.5}$$

where the summation on k runs over all quark flavours. These terms describe the interaction of spin-$\frac{1}{2}$ quarks of mass m and massless spin-1 gluons. The γ matrices satisfy the anticommutation relation $\{\gamma^\mu, \gamma^\nu\}$=$2g^{\mu\nu}$ where $g^{\mu\nu}$ is given by $g^{\mu\nu}$=diag(1,-1,-1,-1). $F^a_{\mu\nu}$ is the field strength tensor derived from the gluon field A^a_μ

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{1.6}$$

$$D_\mu = \partial_\mu - igT^a A^a_\mu \tag{1.7}$$

The indices a, b, c run over the eight colour degree of freedom of the gluon field, f^{abc} are the SU(3) structure constants and T^a the $SU(3)$ generators. It is the third "non-abelian" term on the right-hand side of Eq.(1.6) which distinguishes QCD from QED, giving rise to triplet and quartic gluon self-interactions and ultimately to the property of asymptotic freedom. We will be concerned with the massless approximation, setting $m = 0$ in Eq.(1.5) (in the Standard Model, the mass terms are generated by coupling to the Higgs field and do not enter directly as in Eq.(1.5), though the effect is the same as far as QCD is concerned).

The gauge invariance of the Lagrangian where the field A^a_μ has the freedom of gauge transformation

$$A^a_\mu \longrightarrow A^a_\mu + f^{abc}\theta^b A^c_\mu - \frac{1}{g}\partial_\mu \theta^a, \tag{1.8}$$

with the infinitesimal parameters θ^a, forces us to eliminate two unphysical degrees of freedom of the gauge boson by including a gauge fixing term. One way is to put a constraint on the field A^a_μ , and as a covariant constraint we may choose the Lorentz condition

$$\partial^\mu A^a_\mu = 0. \tag{1.9}$$

It is well known from classical dynamics that the Lagrange multiplier method is the most useful in dealing with a constrained system. According to this method we add the term $\lambda(\partial^\mu A^a_\mu)^2$ to the Lagrangian (1.5) instead of directly imposing the constraint (1.9) on the field equation. It is customary to write $\lambda=\frac{-1}{2\alpha}$. The parameter α is called the gauge parameter. The modified Lagrangian reads

$$\mathcal{L}_{classical} + \frac{-1}{2\alpha}(\partial^\mu A^a_\mu)^2. \tag{1.10}$$

The term added is called the gauge-fixing term ($\mathcal{L}_{gauge-fixing}$).

There is a further problem connected with the existence of the three gluon and four gluon self interactions in the Lagrangian (1.10). Performing a perturbative calculation for the one-loop gauge field contribution to the self-energy part $[\Pi^{ab}_{\mu\nu}]$ for the gauge field A^a_μ, one may show that $\Pi^{ab}_{\mu\nu}$ does not satisfy the requirement of gauge invariance $q^\mu\Pi^{ab}_{\mu\nu}(q)=0$ [2]. Moreover if one calculates the gluon scattering cross section corresponding to the tree diagrams shown in Fig.1.1 , and applies a polarization sum in the final state, one fails to obtain a correct expression for the cross section satisfying unitarity. The above difficulty is related to the fact that we did not properly extract the physical polarization for the gauge field even with the gauge-fixed Lagrangian (1.10). Resolving the difficulty involves the introduction of new fictitious fermionic fields which are called Fadeev-Popov ghosts. The contribution of the Fadeev-Popov ghosts should be added to every gauge field loop diagram in order to obtain a correct result. To work in covariant gauge including the ghost term, we add to the Lagrangian

$$\mathcal{L}_{ghost} = \overline{\eta}^a(-\partial^\mu D^{ab}{}_\mu)\eta^b, \tag{1.11}$$

where the η^a are the ghost fields and the covariant derivatives now include the representation of the η^a in the adjoint representation $(T^a)^{bc}=if^{abc}$. The ghost fields cancel

unphysical degree of freedom which otherwise propagate in covariant gauge.

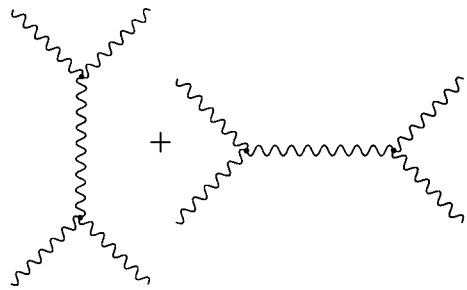

Fig.1.1: Tree diagrams relevant to the elastic scattering of gauge particles

1.4 Path Integrals-Feynman Rules

The Lagrangian which we have proposed for QCD in Eq.(1.5) was derived by extending classical ideas. To describe the behaviour of particles and fields in the real world we must quantize the theory. To do this we can use the path integral formalism [6, 7].

We are aiming to describe the nature of the interaction between particles. The experimentally measured quantity that yields the most information about the dynamics of particle interactions is the cross-section for a process in which two particles scatter off each other. This is related to the scattering amplitude which in turn is closely related to the Green's functions of the theory. Using the path integral formalism we can arrive at a method for relating the Green's functions to the Lagrangian of the theory

Referring to text book [8] we can write down a vacuum to vacuum transition amplitude in the presence of the source $J(x)$,

$$Z[J] = N\int exp[i\int (S(\phi) + \int d^4x J(x)\phi(x))]D\phi, \tag{1.12}$$

where the normalization N is chosen such that $Z[0] = 1$ and $D\phi$ denotes a path integral over all possible functions and with

$$S(\phi) = \int d^4x \mathcal{L}\,, \tag{1.13}$$

the classical action. The Lagrangian ($\mathcal{L}$) generally contains an interaction term

$$\mathcal{L} = \mathcal{L}_0 + g\mathcal{L}_{int}(\phi)\,, \tag{1.14}$$

and the final expression for the transition amplitude $Z[J]$ will be,

$$Z[J] = exp[ig\int d^4x\mathcal{L}_{int}(-i\frac{\delta}{\delta J(x)})]Z_0[J], \tag{1.15}$$

where

$$Z_0[J] = exp[\frac{-i}{2}\int d^4x'\int d^4x J(x')\Delta_F(x'-x)J(x)]. \tag{1.16}$$

$\Delta_F(x'-x)$ is the Feynman propagator and is defined by

$$\Delta_F(x'-x) = \int \frac{d^4p}{2\pi^4}e^{-i\pi(x'-x)}\frac{1}{p^2-m^2+i\epsilon}. \tag{1.17}$$

The prescription $\epsilon\rightarrow 0^+$ displaces the poles at $p=\pm m$ away from the real axis as required by causality. The expression (1.15) is uncalculable and it is due to this that we motivate a perturbative expansion in g.

$$Z[J] = [\sum_{n=0}^{\infty}\frac{i^n g^n}{n!}\int d^4x_1...d^4x_n\mathcal{L}_{int}(-i\frac{\delta}{\delta J(x_1)})...\mathcal{L}_{int}(-i\frac{\delta}{\delta J(x_n)})]Z_0[J]. \tag{1.18}$$

On the other hand, $Z[J]$ can be obtained originally by taking the continuum limit of a set of paths between some initial space time point x_i and a final one x_f, with the interval divided into n intermediate points. Following this procedure we will arrive at functional derivatives

$$\frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)...\delta J(x_n)}\propto i^n < 0, t=+\infty|T[\hat{\phi}_1(x_1)\hat{\phi}_1(x_2)...\hat{\phi}_1(x_n)]|0, t=-\infty >, \tag{1.19}$$

where T denotes the time ordering operator and $\hat{\phi}_1(x_1)$,$\hat{\phi}_1(x_2)$,... are operators with eigenvalues $\phi_1(x_1)$,$\phi_1(x_2)$,....

The vacuum expectation values of the time ordered products of fields precisely define the n-particle Green's function for which we have been searching,

$$\Gamma(x_1,x_2,...,x_n) =< 0|T[\hat{\phi}_1(x_1)\hat{\phi}_1(x_2)...\hat{\phi}_1(x_n)]|0> . \tag{1.20}$$

From (1.18) and using (1.19) we can make calculations of the Green's function at any order in the perturbation expansion and hence obtain the Feynman rules. The Green function Γ is the sum of all Feynman diagrams. All of the information of the theory is

contained in the Green functions.

Working in momentum space (i.e $\partial_\alpha = -ip^\alpha$) we can consider the one particle irreducible (1PI) Green's functions $\Gamma^{n_A,n_F}(p)$ where n_A (n_F) is the number of external gluons (quarks). In practical terms the inverse quark and gluon propagators can be obtained from the free piece $\mathcal{L}_0$ of the QCD Lagrangian. We find for the leading term in the quark 2-point function

$$\Gamma^{(0,2)}_{ab}(p) = -i\delta_{ab}(\not{p} - m) \,, \tag{1.21}$$

where $\not{p} = \gamma_\mu p^\mu$. Similarly the gluon propagator is given by

$$\Delta^2{}_{\{ab,\mu\nu\}}(p) = \delta_{ab}\frac{i}{p^2 + i\epsilon}[-g_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2 + i\epsilon}]. \tag{1.22}$$

The $i\epsilon$ prescription has been used to ensure causality. Without the gauge fixing term it is impossible to define the gluon propagator in a covariant gauge.

Higher-order perturbative corrections to tree-level results in QCD, involve Feynman diagrams containing closed particle loops. These loops translate to give divergent integrals which can be classified into two types *Ultra-violet* (*UV*) divergence (including high momentum regions of momentum space) and *Infra-red* (*IR*) divergences (caused by the divergent behaviour of the integrand as the loop momenta become small).

The propagators are derived from the terms which are bilinear in the fields. The necessary propagators for Feynman rules in momentum space are drawn in Fig.1.2. The other terms (trilinear and quartic in the fields) give us the interaction with the physical vertices being replaced by the corresponding vertex factors which are drawn as well in Fig.1.2. The external fermion legs (straight lines) are given by Dirac spinors and external gluons (spring lines) by the polarization vectors $\epsilon^{\mu(*)}$. Ghosts are represented by dashed lines.

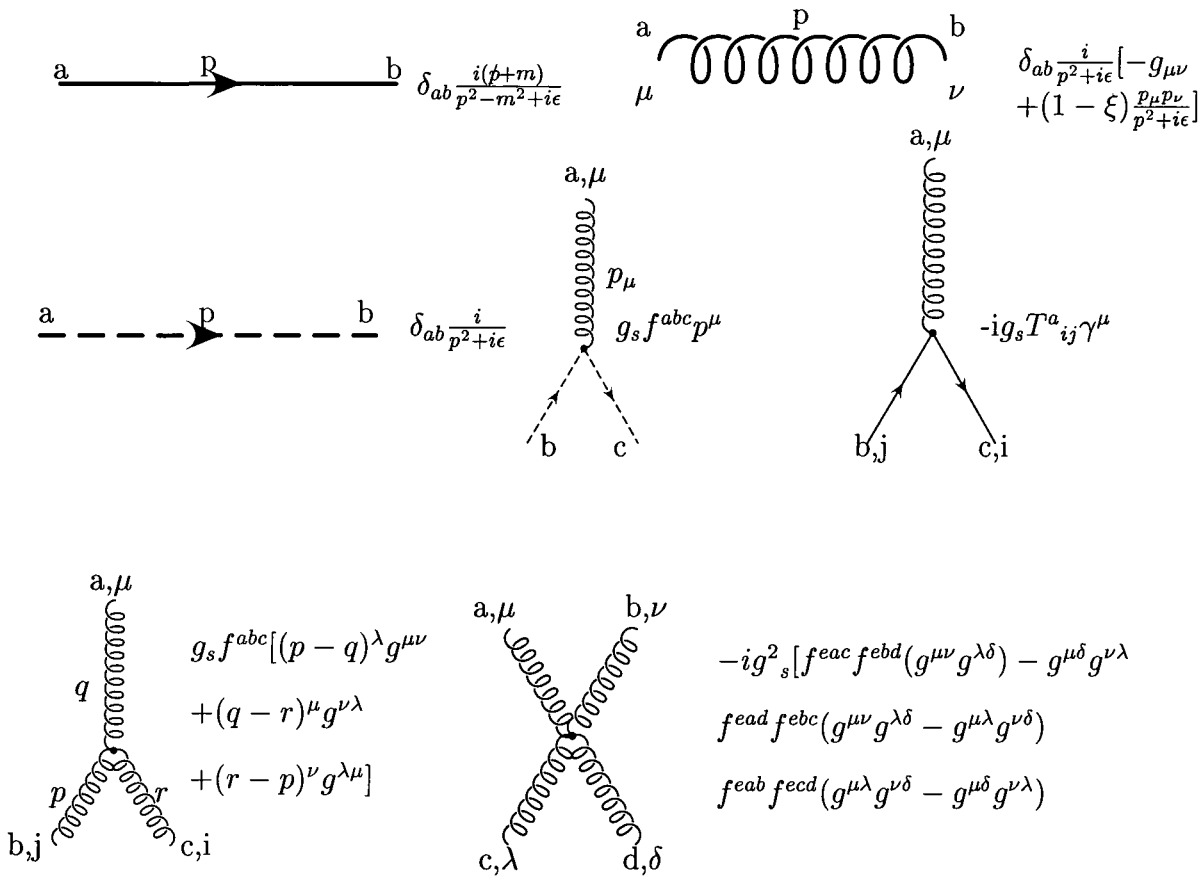

Fig 1.2: The propagator and vertex factors for the QCD Feynman rules. The flow of charge is indicated by arrows.

1.5 Regularization and Renormalization

We saw in the previous section that as soon as we attempt to calculate loops in the perturbation series of the interacting theory, we encounter divergences. We must look for some way to treat these divergences in order to arrive at a finite result. In this case we need to regularize the theory and impose on them some prescription (e.g high momentum cut-off or a modified space time dimension) such that they are finite. The idea is that at the end of the calculation the divergences will have been absorbed, and we will be able to take a physical limit. In the simplest method (cut-off) we impose a large momentum cut-off by hand. This contravenes both translational and gauge invariance, rendering it useless for gauge theories.

It is also possible to regulate the integral by discretising space-time and placing it on a lattice with a spacing which corresponds to the momentum cut-off. This breaks Lorentz and translational invariance although it is the only method to permit a non-perturbative treatment.

Another method, dimensional regularization, has the desirable properties of maintaining Lorentz and gauge invariance and regulating the UV divergences. It treats the number of dimensions as a parameter (D=4-2ϵ) and this enables the integral to become finite by reducing the number, i.e, ϵ>0 and then analytically continuing back to 4 dimensions, all divergences are isolated by the parameter ϵ.

In order to keep the correct mass dimension of the field in the Lagrangian and a dimensionless coupling we are forced to put in a mass scale μ, and rewrite the gauge coupling constant g in the following way

$$g \rightarrow g\mu^{2-\frac{D}{2}}, \tag{1.23}$$

where g is the dimentionless gauge coupling constant and consequently in $D = 4 - 2\epsilon$ space time dimensions we will have

$$g^2 \frac{d^4k}{(2\pi)^4} \rightarrow g^2\mu^{2\epsilon} \frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}}. \tag{1.24}$$

This amounts to changing the volume element. In practice the pole at ϵ =0 always appears in the combination:

$$\frac{\Gamma(1+\epsilon)}{\epsilon}(4\pi)^{\epsilon} = \frac{1}{\epsilon} + ln(4\pi) - \gamma_E + O(\epsilon), \tag{1.25}$$

where Γ is the gamma function and γ_E=0.57721... is the Euler-Mascheroni constant.

There are several equivalent ways to renormalise the theory, two of which are multiplicative renormalization (MR) and the method of counterterms. MR ensures that for every bare (unrenormalized) operator there exists a regulator-dependent multiplicative

factor Z, the *renormalization constant*, that makes the operator independent of regulator when expressed in terms of renormalized quantites. MR involves summing the infinite series of loop diagrams for some fixed number of external lines. This divergent sum is then absorbed into a redefinition of the coupling constant and the mass in the "*bare*" Lagrangian under the assumption that the bare coupling and mass are unmeasurable (infinite) quantities.

In QCD we can redefine the fields and parameters by

$$\begin{aligned} \psi_B &= Z_\psi{}^{\frac{1}{2}}\psi, \quad & A^a_{\mu B} &= Z_A{}^{\frac{1}{2}}A^a_\mu, \quad & \eta_B{}^a &= Z_\eta{}^{\frac{1}{2}}\eta^A \\ g_B &= Z_g g, & m_B^2 &= Z_m m^2, & \xi_B &= Z_A\xi \end{aligned} \tag{1.26}$$

where B denotes the bare operator. A renormalizable theory will require only a finite number of Z's to render it finite to any order. An important property of gauge therories such as QCD or QED is that the underlying gauge invariance guarantees the renormalizability of the theory by reducing the number of renormalization constants required. Specifically only Z_ψ is required for all quark fields and only one Z_A for both the gauge parameter and gluon field. The 3-point ($\overline{\psi}\psi A_\mu$, $A_\nu A_\nu A^\mu$), four-point ($A_\mu A_\nu A^\mu A^\nu$) and ghost-gluon interaction ($\overline{\eta}\eta A_\mu$) involve the same Z_g, hence only a single renormalized coupling is required.

In the second method, counterterms are added directly to the Lagrangian in such a way as to knock out precisely the divergent diagrams. A renormalizable theory will require only a finite number of counterterms to render it finite to any order. Moreover the counterterms are proportional to terms in the original Lagrangian, so adding the two just gives the multiplicative redefinitions of the coupling and the mass as in the first method. We note that a theory is termed renormalizable if all UV divergences can be removed by the introduction of a finite number of renormalization constants. By substituting for bare terms in the Lagrangian we can generate the appropriate counter terms. For example the quark term in Lagrangian (1.5) becomes

$$\overline{\psi}_B(i\gamma^\mu D_\mu)\psi_B = \overline{\psi}_B(i\gamma^\mu\partial_\mu + ig\gamma^\mu T^a A^a_{\mu B})\psi_B = \overline{\psi}(i\not{D})\psi$$

$$+(Z_\psi - 1)\overline{\psi}(i\partial\!\!\!/)\psi + (\mu^\epsilon Z_g Z_\psi Z_A^{\frac{1}{2}} - 1)g\overline{\psi}T^a\gamma^\mu A_\mu^a\psi. \tag{1.27}$$

The first term is exactly the same as in the unrenormalized Lagrangian but now we see two extra *counterterms* arising. Proceeding in a similar manner all propagators and interaction vertices will acquire similar terms.

1.6 Renormalization Group method

The renormalized coupling constant g and mass m depend on the renormalization scale μ at which the subtraction procedure is defined. Writing the dependence explicitly we have

$$g(\mu) = Z_g(\mu)^{-1}g, \quad m(\mu) = Z_m(\mu)^{-1}m\ . \tag{1.28}$$

The renormalized coupling constants $g(\mu)$ and $g(\mu')$ which are obtained through two different subtraction procedures characterized by the renormalization scales μ and μ', repectively, are related to each other by

$$g(\mu') = Z_g(\mu', \mu)g(\mu), \tag{1.29}$$

where the finite renormalization $Z_g(\mu', \mu)$ is given by

$$Z_g(\mu', \mu) = \frac{Z_g(\mu)}{Z_g(\mu')}. \tag{1.30}$$

In the same way we have

$$m(\mu') = Z_m(\mu', \mu)m(\mu), \tag{1.31}$$

where $Z_m(\mu', \mu)$ similarly is defined by

$$Z_m(\mu', \mu) = \left[\frac{Z_m(\mu)}{Z_m(\mu')}\right]^{\frac{1}{2}}\ . \tag{1.32}$$

Let us focus our attention on Eq.(1.29) which define a set of finite renormalization factor $Z_g(\mu', \mu)$ for varying renormalization scales μ' and μ. We regard the finite renormalization (1.29) as a transformation. We can then show that this set of transformations possesses group properties. In fact we can define a product of two elements $Z_g(\mu'', \mu')$ and $Z_g(\mu', \mu)$ as

$$Z_g(\mu'', \mu')Z_g(\mu', \mu), \tag{1.33}$$

which represents the change of $g(\mu)$ through the succesive changes of the scales $\mu \to \mu' \to \mu''$. According to Eq.(1.30) the above product is equal to $Z_g(\mu'', \mu)$ which is nothing but the finite renormalization of $g(\mu)$ caused by the scale change $\mu \to \mu''$. Thus the product $Z_g(\mu'', \mu')Z_g(\mu', \mu)$ also belongs to the set $Z_g(\mu', \mu)$. Morever the inverse of $Z_g(\mu', \mu)$ may be defined by

$$Z_g^{-1}(\mu', \mu) = Z_g(\mu, \mu'), \tag{1.34}$$

as is easily seen, and the identity

$$Z_g(\mu, \mu) = 1\ , \tag{1.35}$$

belongs to the set $Z_g(\mu', \mu)$. Thus this set is a group which is abelian.

In generalization of the above idea, since the Multiplicative Renormalization carries through to the 1PI Green's functions of the theory , Green's functions in two different RS's (denoted by barred and unbarred quantities) would be related to the bare Green's function Γ_B by

$$\Gamma = Z\Gamma_B, \qquad \tilde{\Gamma} = \tilde{Z}\Gamma_B\ , \tag{1.36}$$

where Z and $\tilde{Z}$ are products of different wave function renormalization factors, one for each external leg on the 1PI diagram corresponding to the Green's function. Clearly the two renormalized Green's functions are related by $\tilde{\Gamma}{=}z\Gamma$, where $z{=}\frac{\tilde{Z}}{Z}$, since Z and $\tilde{Z}$ have the same structure of divergence. The z relating different renormalizations of the bare Green's function, would then be products of the Z's, which gives rise to the term *Renormalization Group* as in the above case.

The strength of this extends beyond perturbation theory, holding as an exact property of the theory. This can be expressed analytically in terms of the *Renormalization Group equation* (RGE).

1.7 Renormalization Group equation

The RGE utilizes the independence of the bare 1PI Green's function on the choice of renormalization scheme,

$$\mu \frac{\partial \Gamma_B^{(n_A,n_F)}}{\partial \mu} = 0. \tag{1.37}$$

Consider a 1PI Green's function with n_A, n_F external gluon and quark legs with momentum q_i, p_i repectively. The relation between the bare and renormalized Green's function reads

$$Z_A{}^{n_A} Z_q{}^{n_F} \Gamma_B{}^{(n_A,n_F)}(q_i, p_i, \xi, g) = \Gamma^{(n_A,n_F)}(q_i, p_i, \xi(\mu), g(\mu), \mu). \tag{1.38}$$

Applying $\mu \frac{d}{d\mu}$ to the right-hand side of this equation and using the chain rule give us

$$\begin{aligned} &\mu \frac{d}{d\mu} \Gamma^{(n_A,n_F)}(q_i, p_i, \xi(\mu), g(\mu), \mu) \\ &= (\mu \frac{\partial}{\partial \mu} + \mu \frac{\partial g}{\partial \mu} \frac{\partial}{\partial g} + \mu \frac{\partial \xi}{\partial \mu} \frac{\partial}{\partial \xi}) \Gamma^{(n_A,n_F)}(q_i, p_i, \xi(\mu), g(\mu), \mu). \end{aligned} \tag{1.39}$$

Writing $Z_A^{n_A} = e^{n_A ln Z_A}$ and $Z_q{}^{n_F} = e^{n_F ln Z_q}$ it is easy to get the following relation for the renormalized Green's function

$$(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - n_A \gamma_A(g) - n_F \gamma_F(g) + \delta(g) \frac{\partial}{\partial \xi}) \Gamma^{(n_A,n_F)}(q_i, p_i, \xi(\mu), g(\mu), \mu) = 0 \ , \tag{1.40}$$

where $\beta(g) = \mu \frac{\partial g}{\partial \mu}$, $\gamma_A(g) = \frac{\mu}{Z_A} \frac{\partial Z_A}{\partial \mu}$, $\gamma_F(g) = \frac{\mu}{Z_g} \frac{\partial Z_q}{\partial \mu}$, $\delta(g) = \mu \frac{\partial \xi}{\partial \mu}$. Eq.(1.40) is known as the Renormalization Group Equation (RGE).

We shall see a very important property due to the renormalization group namely that of a running coupling in a Green's function. For simplicity we consider a dimensionless Green's function with n external legs and suppress the dependence on the gauge parameter, $\Gamma^{(n)}(p_i, g(\mu), \mu)$. We will now use Eq.(1.39) in this simple case to derive the way the $\Gamma^{(n)}$-function transforms under a scaling of momenta. Let us scale the momenta such that

$$p_i \to p_i e^t \ . \tag{1.41}$$

Using dimensional arguments and the fact that $\Gamma^{(n)}$ is Lorentz invariant we have

$$\Gamma^{(n)}(e^t p_i, g(\mu), \mu) = \mu^D f(\frac{p_i \cdot p_j}{\mu^2} e^{2t}, g), \tag{1.42}$$

where D is the mass dimension of $\Gamma^{(n)}$. This, in turn, implies that $\Gamma^{(n)}$ satisfies the differential equation

$$(\mu\frac{\partial}{\partial\mu}+\frac{\partial}{\partial t}-D)\Gamma^{(n)}(e^t p_i, g(\mu),\mu)=0, \tag{1.43}$$

which, using RGE (1.40), gives

$$(-\frac{\partial}{\partial t}+\beta(g)\frac{\partial}{\partial g}+D-n\gamma(g))\Gamma^{(n)}(e^t p_i, g(\mu),\mu)=0. \tag{1.44}$$

This has the solution

$$\Gamma^{(n)}(e^t p_i, g(\mu),\mu)=exp[tD-n\int_0^t dt'\gamma(g(t'))]\Gamma^{(n)}(p_i,\bar{g}(t),\mu), \tag{1.45}$$

where $exp[tD]$ is known as the canonical or engineering dimension. The extra term in the exponent is the anomalous dimension. Note that we have introduced a running coupling constant

$$\frac{\partial\bar{g}(t)}{\partial t}=\beta(\bar{g}), \tag{1.46}$$

which enables the rewriting of the anomalous dimension as

$$exp[-n\int_g^{\bar{g}} d\bar{g}\frac{\gamma(\bar{g})}{\beta(\bar{g})}]. \tag{1.47}$$

Knowing β and γ from the theory enables the evaluation of the momentum dependence of the Green's function. These functions are at least theoretically calculable in perturbation theory.

We derive the running coupling, $\bar{g}$, equivalently in another way. In the particular case that $\Gamma^{(n)}$ is dimensionless we may write

$$\Gamma^{(n)}(p_i e^t, g(\mu),\mu)=\mathcal{F}(\frac{p_i\cdot p_j}{\mu^2}e^{2t}, g(\mu))\ , \tag{1.48}$$

$$\Gamma^{(n)}(p_i, g(\mu e^{-t}),\mu e^{-t})=\mathcal{F}(\frac{p_i\cdot p_j}{\mu^2}e^{2t}, g(\mu))\ , \tag{1.49}$$

where Eq.(1.40) then gives

$$(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g})\Gamma^{(n)}(e^t p_i, g(\mu),\mu)=0\ . \tag{1.50}$$

Changing variable to $\lambda = \mu e^{-t}$, and utilizing the equality of Eq.(1.48) and Eq.(1.49) we can write

$$\frac{d}{dt}\Gamma^{(n)}(p_i e^t, g(\mu), \mu) = (-\lambda\frac{\partial}{\partial\lambda} + \mu\frac{\partial}{\partial\mu})\Gamma^{(n)}(p_i e^t, g(\mu), \mu). \tag{1.51}$$

Combining with Eq.(1.50) to eleminate the $\frac{\partial}{\partial\mu}$ term we arrive at

$$(-\frac{\partial}{\partial\lambda} + \beta(g)\frac{\partial}{\partial g})\Gamma^{(n)}(p_i e^t, g(\mu), \mu) = 0. \tag{1.52}$$

This is a homogeneous partial differential equation of order one and may be solved by introducing the *running coupling constant* $\bar{g}(\lambda)$ defined at a renormalization scale λ,

$$\lambda\frac{\partial\bar{g}}{\partial\lambda} = \beta(\bar{g}(\lambda)), \qquad \bar{g}(\lambda = 0) = g. \tag{1.53}$$

Given the change of variable $\lambda = \mu e^{-t}$, equations (1.53) and (1.46) are equivalent. These equations show that the way in which $\bar{g}(t)$ runs with the momentum is entirely governed by the β-function of the theory, and so by studying the β-function we can discover in which momenta region(s) the coupling becomes large.

1.8 The Renormalization Group β-function equation

From Eq.(1.53) we see that the RGE represents a powerful constraint on the renormalized Green function of any quantum field theory, yielding important consequences. Integrating this equation, we have

$$t = \int_g^{\bar{g}} \frac{dx}{\beta(x)}, \tag{1.54}$$

where

$$g(\mu) = g, \quad g(Q) = \tilde{g}, \quad t = ln(\frac{Q}{\mu}), \tag{1.55}$$

from which we can get a transcendental equation for $g(\mu)$.

Since $\Gamma^{(n)}$ is dimensionless all of the scale dependence of $\Gamma^{(n)}$ enters through the running of the coupling constant $g(Q)$. In practical calculations it has become usual to define the strong coupling constant

$$\alpha_s = \frac{g^2}{4\pi} , \tag{1.56}$$

and as a result we obtain a slightly altered β-function

$$\mu^2 \frac{\partial \alpha_s(\mu^2)}{\partial \mu^2} = \beta(\alpha_s) = -\beta_0 {\alpha_s}^2 - \beta_1 {\alpha_s}^3 - \beta_2 {\alpha_s}^4 + ... \tag{1.57}$$

where $\beta_0 = \frac{11C_A - 2N_f}{6\pi^2}$, $\beta_1 = \frac{17{C_A}^2 - 5C_A N_F - 3C_F N_f}{24\pi^3}$ or in the other convention with $a = \frac{\alpha_s}{\pi}$,

$$\mu \frac{\partial a}{\partial \mu} = \beta(a) = -ba^2(1 + ca + c_2 a^2 + ... + c_k a^k + ...) . \tag{1.58}$$

where

$$b = \frac{11C_A - 2N_f}{6}, \quad c = \left[\frac{7}{8}\frac{C_A^2}{b} - \frac{11}{8}\frac{C_A C_F}{b} + \frac{5}{4}C_A + \frac{3}{4}C_F\right] , \tag{1.59}$$

are two universal coefficients , with $C_A = N$ and $C_F = \frac{N^2-1}{2N}$ as eigenvalues of casimir operators in $SU(N)$ symmetry group and N_f the number of flavours. For QCD we see that providing $N_f \leq 16$, $\beta_0 > 0$, then the coupling will decrease with increasing energy. This is the much desired property of asymptotic freedom and ultimately results from the non-abelian gluon self-interaction.

The importance of the β-fuction is in determining the behaviour of the coupling constant. Through its dependence on renormalization point it also has a role to play in the labelling of renormalization schemes. In turning the μ-dependence of the β-function to our advantage we find that choosing a form for the β-function and selecting a renormalization point, enables us to specify a unique Renormalization Scheme (RS).

For many perturbative applications the β-function is truncated at a fixed order. For example if we take the β-function truncated to its one-loop form, we have

$$\beta(\alpha_s) = -ba^2 \tag{1.60}$$

where

$$a = \frac{\alpha_s}{\pi}, \quad b = \pi^2 \beta_0 \, . \tag{1.61}$$

Integrating Eq.(1.58) gives

$$\frac{1}{a(\mu_0)} - bln(\mu_0) = \frac{1}{a(\mu)} - bln(\mu) = -bln(\Lambda) \tag{1.62}$$

or

$$a(\mu) = \frac{1}{bln(\frac{\mu}{\Lambda})} \, , \tag{1.63}$$

where the integration constant Λ is a fundamental constant of the theory to be determined from experiment. It can be naively interpreted as the scale at which the β-function would diverge if extrapolated outside the perturbative domain. More quantatively, it indicates the order of magnitude of the scale at which $\alpha_s(Q^2)$ becomes strong.

In the next chapter we consider further the renormalization group β-function equation and introduce some frequently used renormalization schemes.

Chapter 2

Scheme dependence of perturbative series

2.1 Review of the Renormalization Scheme

The process of renormalization as described in the previous chapter is not unique. In subtracting each UV divergence we are at liberty to choose the finite remainder, providing we remove the infinity in a consistent manner. Consider a generic QCD observable that can always if necessary, be divided by a constant and raised to a suitable power, to arrive at a perturbation series of the form

$$R(a) = a + r_1 a^2 + r_2 a^3 + ...r_n a^{n+1} + ... \,. \tag{2.1}$$

We shall see that termination of the series at any given order will result in a residual dependence on the renormalization procedure being picked up. There will be a dependence on the renormalization scale μ (i.e the scale at which the theory is renormalized), in addition to the finite subtraction dependence. Together these constitute what is known as the *Renormalization Scheme*. Here only the LO (tree-level, leading order) perturbative coefficient is renormalization scheme invariant. Already, at next-to-leading order, NLO, one has a renormalization scheme dependence problem. In higher orders, the subtraction procedure can presumably be chosen to be the same, but an entirely new arbitrary constant will be introduced at each order.

The dependence on these arbitrary constants is obviously unphysical and since the all-orders $R(a)$ should not depend on any arbitrary parameters, it is clear that the renormalization scheme dependence must cancel in an all-orders calculation.

This chapter will discuss the general problem of renormalization scheme (RS) dependence.

2.2 Parametrizing the Renormalization Scheme dependence

In Eq.(2.1) both the coupling and the perturbation series coefficients beyond leading order are dependent on the renormalization scheme used. The problem of parametrizing RS dependence is simplified by three restrictions [9] :1) The restriction to physical quantities, 2) The restriction to massless theories ,and 3) The restriction to perturbation theory.

Consider two RS's as different if they correspond to different definitions of the couplant ($a \equiv \frac{\alpha_s}{\pi}$), i.e., different definitions of the finite part of the renormalization constant Z in the relation $a = Za_{bare}$. The restriction to perturbation theory is important. It allows one to assume that anything and everything can be expressed as a power series in a (in whatever RS). In particular, the coupling $\bar{a}$ in some general RS must be expressible as a power series in the couplant a of some "bare" scheme, i.e.,

$$\bar{a} = a(1 + \nu_1 a + \nu_2 a^2 + ...) \, . \tag{2.2}$$

It seems natural to assume that each coefficient ν_1, ν_2, ... corresponds to a distinct degree of freedom in the specification of one RS in terms of another.

Before discussing further how to parametrize the scheme dependence, we need to

define a parameter τ as

$$\tau = bln\left(\frac{\mu}{\tilde{\Lambda}_{RS}}\right) , \tag{2.3}$$

which is the first RS parameter. $\tilde{\Lambda}_{RS}$ is related to conventional Λ parameter as defined by Buras et al, [1,10] by

$$\tilde{\Lambda}_{RS} = \Lambda_{RS}\left(\frac{2c}{b}\right)^{-\frac{c}{b}} . \tag{2.4}$$

Even though the ν_1, ν_2, ... parameters have the correct number of degrees of freedom to label the scheme, we shall find it advantageous to exchange them for a different set in which τ replaces ν_1. Rewriting the relationship of Eq.(2.2) equivalently using the β-function defined in each RS we get

$$\beta(a) = \frac{da}{d\bar{a}}\bar{\beta}(\bar{a}) , \tag{2.5}$$

where $\beta(a)$ is defined by Eq.(1.58). The β-function has a scheme-dependent expansion. Whilst $c_1 = c$ is universal, the subsequent c_n β-function coefficients can then each be exchanged for the corresponding ν_n to label the scheme. Indeed, it was shown in ref [9] that one may consistently use the parameters τ,c_2,c_3,... to label the renormalization scheme. In the conventional approach when retaining terms up to and including $r_n\ a^{n+1}$ in Eq.(2.1) one truncates the β-function of Eq.(1.58) and retains terms up to and including $c_n a^{n+2}$. On integrating up the truncated Eq.(1.58) one can define $a^{(n)}(\tau, c_2, ...c_n)$ and correspondingly one finds for consistency the dependences $r_1(\tau)$, $r_2(\tau, c_2)$, $r_3(\tau, c_2, c_3)$, ... and $r_n(\tau, c_2, c_3, ...c_n)$. In this way, the n^{th} order truncated approximant is also labelled by the scheme variables $R^{(n)}(\tau, c_2, ...c_n)$. Of course when summed to all orders this dependence must cancel and a formally RS-independent sum be obtained.

Furthermore, we can see how to make a scheme-invariant quantity from the scheme label parameter, τ. At NLO we can write the truncated Eq.(2.1) as

$$R^{(1)}(\tau) = a(\tau) + r_1 a(\tau)^2, \tag{2.6}$$

where $a(\tau)$ is defined by integrating up the NLO truncated β-function

$$\frac{da}{d\tau} = -a^2(1+ca). \tag{2.7}$$

Taking the boundary condition at $a(\tau=0)=\infty$ we obtain

$$\tau = \frac{1}{a(\tau)} + c\ln\left(\frac{ca(\tau)}{(1+ca(\tau))}\right) \equiv F(a). \tag{2.8}$$

where we define the function F for later use, so that $a^{(1)}(\tau)=F^{-1}(\tau)$.

To determine the dependence of r_1 on τ, we first relate two r_1 values from different schemes by substituting Eq.(2.2) into Eq.(2.5) and equating coefficients, to get

$$r_1 = \overline{r_1} + \nu_1 \ , \tag{2.9}$$

Similarly we can use Eq.(2.8) to find the difference of two τ's in different schemes.

$$\tau - \bar{\tau} = \frac{1}{a} + c\ln\left(\frac{ca}{(1+ca)}\right) + O(a) - \frac{1}{\bar{a}} - c\ln\left(\frac{c\bar{a}}{(1+c\bar{a})}\right) + O(\bar{a}) \ , \tag{2.10}$$

where $O(a)$ and $O(a')$ terms reflect contributions beyond NLO in β-function.
Since this should hold for all values of μ we can take the $\mu\to\infty$ limit and utilizing asymptotic freedom ($a(\mu=\infty)=0$) and get

$$\tau - \bar{\tau} = \nu_1 \tag{2.11}$$

Eliminating ν_1 between Eqs.(2.9) and (2.11) gives

$$\tau - r_1(\tau) = \bar{\tau} - r_1(\bar{\tau}) \ . \tag{2.12}$$

This implies that

$$r_1(\tau) = \tau + r_1(0) \ , \tag{2.13}$$

and from there we can identify the RS-invariant combination [9]

$$\rho_0 = \tau - r_1(\tau) \ . \tag{2.14}$$

Having discussed how to parameterise different schemes, in the following we briefly review some important schemes.

2.3 Preliminary remarks on renormalization schemes

Given an explicit expression for the quark self-energy to one-loop order , i.e.

$$\sigma(p^2) = 2g_0^2 C_F \int \frac{d^4l}{(2\pi)^4} \frac{1}{(p+l)^2 l^2} , \tag{2.15}$$

or the regularized form

$$\sigma(p^2) = -\frac{g_0^2}{(4\pi)^2} C_F \left(\frac{1}{\epsilon} - \gamma_E + ln(4\pi) + 1 - ln(\frac{-p^2}{\mu^2}) \right) + O(\epsilon) , \tag{2.16}$$

where g_0 is the bare coupling. We shall make some preliminary arguments about various renormalization schemes by using this concrete example. It is important to note here that the way of eliminating divergences in perturbation theory is not unique because there exists an ambiguity in defining the divergent piece of the Green function. The prescription used in subtracting divergences in Green functions as discussed before, is called the *Renormalization Scheme*. Two different renormalization schemes are always connected by a finite renormalization.

Returning to the one-loop quark self-energy part $\sigma(p^2)$ in Eq.(2.15) we can explain renormalization schemes with the help of it [2]. With $m = 0$ we can define the quark propagator

$$\tilde{S}_{ij}(p) = -\frac{\delta_{ij}}{\not{p}} \frac{1}{1+\sigma(p^2)} , \tag{2.17}$$

We renormalize the quark propagator $\tilde{S}_{ij}(p)$ by a multiplicative factor Z_2 (the quark-field renormalization constant),

$$\tilde{S}_{Rij}(p) = {Z_2}^{-1} \tilde{S}_{ij}(p) , \tag{2.18}$$

where $\tilde{S}_{Rij}(p)$ is the renormalized (finite) quark propagator. We expand Z_2 in powers of g_0 and write

$$Z_2 = 1 - z_2 + O(g_0^2) , \tag{2.19}$$

with the z_2 the term of order g_0^2 which is assumed to be divergent. Substituting Eq.(2.19) for Z_2 in Eq.(2.18) and using Eq.(2.17) we obtain to order g_0^2

$$\tilde{S}_{Rij}(p) = -\frac{\delta_{ij}}{\not{p}} \frac{1}{1+\sigma(p^2)-z_2} \tag{2.20}$$

where we keep only the terms up to order g_0^2 in the denominator. Since $\tilde{S}_{Rij}(p)$ is the renormalized propagator, it should be free of divergences and hence $\sigma(p^2) - z_2$ has to be finite. This requirement determines the constant z_2 up to a finite additive constant. In order to fix this arbitrary finite constant in z_2, we need an additional requirement which constitutes a renormalization scheme (prescription). There are a variety of such renormalization schemes. In the following we refer briefly to these different kinds of schemes.

2.3.1 On-shell subtraction

The renormalization constant Z_2 is determined on the quark mass shell i.e. by the condition

$$\tilde{S}_{Rij}(p) = -\frac{\delta_{ij}}{m - \not{p}}, \qquad for \quad \not{p} \sim m. \tag{2.21}$$

This is the traditional renormalization prescription employed in QED and is the most natural for obtaining physical quantities such as S-matrices. For the present case $m = 0$ and $z_2 = \sigma(0)$. Unfortunately for massless quarks a mass singularity develops in $\sigma(p^2)$ and $\sigma(0)$ is not well-defined in the present example.

2.3.2 Off-shell subtraction (Momentum-space subtraction) (MOM)

At an unphysical (off-shell) value of p^2, say $p^2 = -\lambda^2 < 0$, we require that $\tilde{S}_{Rij}(p)$ be of the form of the free (massless) propagator

$$\tilde{S}_{Rij}(p) = -\frac{\delta_{ij}}{-\not{p}} \qquad for \quad \not{p}^2 \sim \lambda^2 \tag{2.22}$$

with this renormalization condition [23, 24] we determine z_2 such that

$$z_2 = \sigma(-\lambda^2) = -\frac{g_0^2}{(4\pi)^2} C_F \left(\frac{1}{\epsilon} - \gamma_E + 1 - ln(\frac{-\lambda^2}{4\pi\mu^2}) \right). \tag{2.23}$$

Hence the renormalised propagators reads

$$\tilde{S}_{Rij}(p) = -\frac{\delta_{ij}}{-\not{p}} \left(1 + \frac{g_0^2}{(4\pi)^2} C_F ln(\frac{-p^2}{\lambda^2}) \right)^{-1}. \tag{2.24}$$

This gives rise to the more complicated counterterm, a further ambiguity lies in the choice of which vertex to take in the definition of the coupling (for example we could have taken the triple gluon vertex). Lastly we note that due to the structure of Z_g in all MOM schemes, the β-function is gauge dependent. As a result of these aspects, not last of which is the complexity of the counterterms, the MOM schemes are not commonly used in perturbative calculations.

2.3.3 Minimal Subtraction (MS)

This renormalization condition is due to 't Hooft [11] and is specific to dimensional regularization. In the Minimal Subtraction scheme we eliminate only the pole term $\frac{1}{\epsilon}$ in the dimensionally regularized expression of the Green functions. As this scheme is the most economical and has some further advantages, it has been frequently used in many applications of QCD (and also other gauge-field therories). With this renormalization condition we have

$$z_2 = -\frac{g_0^2}{(4\pi)^2} C_F (\frac{1}{\epsilon}) \,. \tag{2.25}$$

The renormalised propagator in this scheme is

$$\tilde{S}_{Rij}(p) = -\frac{\delta_{ij}}{\not{p}} \left\{ 1 + \frac{g_0^2}{(4\pi)^2} C_F \left(\gamma_E - 1 + ln(\frac{-p^2}{4\pi\mu^2}) \right) \right\}^{-1} . \tag{2.26}$$

Thus in the minimal subtraction scheme the renormalization constants acquire the simplest expression while the renormalized Green functions have rather complicated forms. Since the renormalization constants in this scheme are independent of mass parameters in the theory, we shall have great facility in defining the renormalization group functions. Note here that the expression of Eq.(2.26) may be converted to that of Eq.(2.24) in MOM scheme by setting

$$\lambda^2 = 4\pi e^{1-\gamma_E} \mu^2 \,. \tag{2.27}$$

2.3.4 Modified Minimal Subtraction ($\overline{MS}$)

In the expression for $\sigma(p^2)$ in Eq.(2.16) the pole term is accompanied by the natural constant γ_E and $ln(4\pi)$ in the combination

$$\frac{1}{\epsilon} - \gamma_E + ln(4\pi) \,. \tag{2.28}$$

The appearance of this combination is a phenomenon peculiar to dimensional regularization with the conventions described in Section 1.5. It can be shown that the combination (2.28) always appears in any calculations to one-loop order. Hence it is more convenient to eliminate, in the renormalization process, the whole of Eq.(2.28) instead of eliminating only the term $\frac{1}{\epsilon}$ as in the MS scheme. This is the modified minimal subtraction ($\overline{MS}$) scheme [1] which is frequently used in the definition of the QCD coupling constant and also in other applications of gauge field theories. The renormalization constant z_2 in the $\overline{MS}$ scheme takes the following form,

$$z_2 = -\frac{g_0^2}{(4\pi)^2} C_F \left(\frac{1}{\epsilon} - \gamma + ln(4\pi) \right) . \tag{2.29}$$

The renormalized propagator reads

$$\tilde{S}_{Rij}(p) = -\frac{\delta_{ij}}{\not{p}} \left\{ 1 + \frac{g_0^2}{(4\pi)^2} C_F \left(-1 + ln(\frac{-p^2}{\mu^2}) \right) \right\}^{-1} . \tag{2.30}$$

The $\overline{MS}$ scheme shares many advantages with the MS scheme and at the same time offers a significantly more compact expression for the renormalised propagator. Through the use of dimensional regularisation and subsequent minimal subtraction, no physical meaning is attributed to the renormalization scale, μ, leaving it an entirely free, although artifical parameter. Its algebraic simplicity and ease of implementation have resulted in the widespread use of the $\overline{MS}$ renormalization scheme in perturbative QCD calculations.

As we have seen, the above four different renormalization schemes provide different forms for the renormalised propagator. In general the form of the Green functions varies from scheme to scheme.

After introducing these different renormalization schemes, we are now in a position to consider different approaches to the solution of the scheme dependence problem for QCD observables.

2.4 Various solutions of Scheme dependence problem

Due to the strong dependence of the QCD coupling on the renormalization scheme, there have been a number attempts to resolve the ambiguities inherent in fixed order perturbative calculations.

In $NNLO$ and higher one can show [9, 12, 13] that for a given value of ρ_0 there exists a choice of τ, c_2, ..., c_n such that $R^{(n)}$ has any desired positive value. Since $R(a)$ in Eq.(2.1) is a function of a single scale, Q, we can write ρ_0 as

$$\rho_0 = \tau - r_1(\tau) \equiv b \ln(\frac{Q}{\Lambda_R}) \,, \tag{2.31}$$

where to maintain dimensionality we have defined an observable but RS-invariant quantity, Λ_R. Eq.(2.31) specifies precisely how the NLO coefficient depends on the scale. Reconsidering Λ_R, we note that it is also possible to relate this observable dependent quantity with the dimensional transmutation parameter $\tilde{\Lambda}_{RS}$ that is universal but renormalization scheme (RS)-dependent. By rearranging Eq.(2.31) and combining it with Eq.(2.3) we will have

$$\Lambda_R = \tilde{\Lambda}_{RS} exp\left(\frac{{r_1}^{RS}(\mu = Q)}{b}\right). \tag{2.32}$$

For any $\tilde{\Lambda}_{RS}$ we can choose a sequence of schemes such that $R^{(2)}$=$R^{(3)}$=..=$R^{(n)}$= $R^{(exp)}$, the experimentally measured value.

Various "solutions" of the scheme dependence problem, i.e. motivation for particular choices of τ, c_2,..., c_n have been proposed. In the remaining parts of this section we shall review a number of these proposals.

2.4.1 The Physical Scale

The approach which is probably most often taken in NLO calculation is to dispose of the renormalization scale dependence by choosing it to be close to the physical scale of the problem, $\mu \simeq Q$. The motivation for this viewpoint is the fact that perturbative coefficients in higher orders will be polynomials in $\ln(\frac{\mu}{Q})$ so

$$r_n = \sum_{l=0}^{n} K_{nl} \left(b \ln(\frac{\mu}{Q}) \right)^l . \tag{2.33}$$

If predictions in the vicinity of $\mu = Q$ are strongly μ-dependent then this is supposedly an indication that the perturbation series is intrinsically badly behaved. Thus by setting $\mu = Q$ in Eq.(2.33) one avoids unnecessarily large logarithms. This overlooks the fact that the NLO renormalization scheme dependence is not completely given by the dependence on the renormalization scale. Since $r_n(\tau, c_2, ..., c_n)$ where $\tau = b \ln(\frac{\mu}{\Lambda_{RS}})$, the coefficient K_{nl} above will also depend on $\tilde{\Lambda}_{RS}$. However we can rewrite Eq.(2.33) in the form

$$r_n(\tau, c_2, ..., c_n) = \sum_{l=0}^{n} \tilde{K}_{nl} (\tau - \rho_0)^l . \tag{2.34}$$

To avoid unnecessarily large terms we should clearly choose $\tau = \rho_0$. We notice that

$$\tau - \rho_0 = [bln(\frac{\mu}{Q}) + r_1{}^{RS}(Q)] = bln[\frac{\mu}{Q} e^{\frac{r_1{}^{RS}(Q)}{b}}] , \tag{2.35}$$

so we can write Eq.(2.34) as

$$r_n(\tau, c_2, ..., c_n) = \sum_{l=0}^{n} \tilde{K}_{nl} \left[bln \left(\frac{\mu}{Q} e^{\frac{r_1{}^{RS}(Q)}{b}} \right) \right]^l . \tag{2.36}$$

The coefficients $\tilde{K}_{nl}$ depend on the perturbation coefficient r_2, r_3,..., r_k and other renormalization scheme parameters c_2, c_3, ..., c_k but crucially do not depend on the NLO renormalization scheme choice whereas in Eq.(2.33) the K_{nl} depend on $r_1^{RS}(Q)$ and so have a hidden dependence on the NLO renormalization scheme choice which is customarily ignored in the usual "physical scale" argument.

Choosing μ so as to avoid large logarithms in Eq.(2.36) corresponds to $\mu = Qe^{\frac{-r_1{}^{RS}(Q)}{b}}$, we conclude that a modified "avoidance of unnecessarily large logarithms" physical

scale argument which correctly labels the NLO renormalization scheme dependence actually leads to the so-called Effective Charge scheme, $\tau = \rho_0$ that we will introduce in subsection 2.4.4.

2.4.2 The Principle of Minimum Sensitivity (PMS)

The principle of Minimum Sensitivity (PMS) approach attempts to resolve the renormalization scheme dependence problem by exploiting the fundamental notion of renormalization group invariance of physical quantities. It was suggested by P.M.Stevenson [9], that, since the final result of a physical calculation should be independent of arbitrary, unphysical parameters, one should choose such parameters by minimising the dependence on them of the truncated approximant to the quantity being calculated. In fact since the exact all-orders result is independent of the renormalization scheme parameters, one should choose the n^{th}-order approximation to mimic this property and to be as insensitive as possible to the chosen value of these unphysical parameters. That is, one arranges

$$\frac{dR^{(n)}}{d(RS)} \mid_{RS=optimal\ RS} = 0. \tag{2.37}$$

Mathematically PMS is formulated as a variational principle and we can arrange this as

$$\frac{\partial R^{(n)}}{\partial \tau} \mid_{\tau=\bar{\tau}} = \frac{\partial R^{(n)}}{\partial c_2} \mid_{c_2=\overline{c_2}} = \ldots = \frac{dR^{(n)}}{\partial c_n} \mid_{c_n=\overline{c_n}} = 0. \tag{2.38}$$

The PMS scheme is then specified by $\bar{\tau}$, $\overline{c_2}$, ..., $\overline{c_n}$.

To give an example of PMS we consider NLO. This corresponds to minimising the τ dependence . We solve

$$\frac{\partial R^{(n)}}{\partial \tau} \mid_{\tau=\bar{\tau}} = \left(\frac{\partial}{\partial \tau} \mid_{\bar{a}} + \frac{\beta(a)}{b} \frac{\partial}{\partial a} \right) R^{(n)} = 0. \tag{2.39}$$

This leads to

$$\bar{a}^2 \frac{\partial r_1}{\partial \tau} - \bar{a}^2 (1 + c\bar{a})(1 + 2r_1\bar{a}) = 0, \tag{2.40}$$

where the barred quantities are evaluated with the PMS NLO scheme parameters, $\tau = \bar{\tau}$. Eq.(2.31) enforces us with $\frac{\partial r_1}{\partial \tau} = 1$ and the $O(a^2)$ terms must cancel for the formal consistency of perturbation theory. The PMS criterion requires the remaining terms of Eq.(2.40) to vanish at $\tau = \bar{\tau}$,

$$2\bar{r}_1(1 + c\bar{a}) + c = 0, \tag{2.41}$$

where $\bar{r}_1 = r_1(\bar{\tau})$, $\bar{a} = a(\bar{\tau})$. Eliminating $\bar{r}_1$ in Eq.(2.6), using Eq.(2.41) we then obtain the solution in terms of $\bar{a}$

$$R^{(1)}_{PMS} = \frac{\bar{a}(1 + \frac{1}{2}c\bar{a})}{(1 + c\bar{a})}. \tag{2.42}$$

Substituting Eq.(2.41) into Eq.(2.14) we are left with the corresponding transcendental equation

$$\rho_0 = \frac{1}{\bar{a}} + c \ln\left[\frac{c\bar{a}}{1 + c\bar{a}}\right] + \frac{1}{2}\frac{c}{1 + c\bar{a}}. \tag{2.43}$$

Solving this transcendental equation gives the PMS coupling which can then be used with Eq.(2.42) to give the NLO estimate for R. We see that $\bar{a}$ and hence $R^{(1)}_{PMS}$ is a renormalization scheme invariant. The only possible scheme dependence entering $R^{(1)}(\bar{a})$ is through the couplant and from Eq.(2.43) we can see that the couplant is defined only through renormalization scheme invariants. This is also true for higher-order optimized approximants. If $R(a)$ and all derivatives are monotonic then the perturbative approximant is irredeemably ambiguous since there is no objective means for choosing μ at which to evaluate the coupling. This happens in lowest order $R(a) = a$ which therefore gives at best semi-quantitative results.

PMS correctly attempts to optimize μ and $\tilde{\Lambda}_{RS}$, both of which are required at NLO. The disadvantages of the PMS method are that the coupling and the β-function are unphysical quantities and that given the complex nature of the coupled equations which must be solved, it is unclear whether their all-orders versions are defined.

2.4.3 BLM Scale Fixing

Brodsky, Lepage and Mackenzie (BLM) [14], were motivated by physical considerations in QED, where the photon self-energy corrections are absorbed into the coupling constant by an appropriate choice of scale. They attempted to reproduce this key idea within the context of QCD, and assumed that there exists an optimal scale μ at every order. They were looking for a well-defined renormalization scheme because of the lack of a boundary condition value for α_s, in contrast to the low energy value of $\alpha_{QED}=\frac{1}{137.035...}$ in the QED case. If there are not gluon-gluon interactions, the optimal scale is obtained by the requirement that all light quark vacuum polarization corrections are absorbed within the coupling.

The BLM proposal is to select the scale μ so that the NLO coefficient r_1 is independent of the number of flavours N_f. The flavour-dependence of r_1 is then absorbed into the coupling α_s. The linearly N_f-dependent part of r_1 appears in the calculated Feynman diagrammatics due to the insertion of a one-loop fermion bubble correction in the photon propagator. The generalisation of this property in QCD is termed "naive nonabelianization" where there will be additional gluon and gluon loop corrections relevant in the running of α_s. Attempts to extend BLM to higher orders have been made [15], but for simplicity we consider the NLO case. Consider a physical observable , defined in two RS's, one the BLM scheme and the other arbitrary. This gives

$$R = a + r_1 a^2, \quad R^{BLM} = a_{BLM} + {r_1}^{BLM} a_{BLM}^2. \tag{2.44}$$

We can separate linearly N_f-dependent and N_f-independent contributions for both r_1 and the first β-function coefficient, b such that

$$r_1 = {r_1}^{(1)} N_f + {r_1}^{(0)}, \quad b = b_{(1)} N_f + b_{(0)}. \tag{2.45}$$

The two couplings can be related via Eq.(2.9) and Eq.(2.12) which give us $a_{BLM} = \frac{a}{1+b\ln(\frac{\mu_{BLM}}{\mu})}$. Using Eq.(2.45) the BLM perturbative coefficient will then be

$${r_1}^{BLM} = {r_1}^{(1)} N_f + {r_1}^{(0)} + (b_{(1)} N_f + b_{(0)}) \ln(\frac{\mu_{BLM}}{\mu}), \tag{2.46}$$

where μ_{BLM} and μ are the renormalization scales in the respective schemes. Equating N_f independent and dependent parts we obtain

$$\mu_{BLM} = \mu exp(\frac{-r_1{}^{(1)}}{b_{(1)}}), \quad r_1{}^{BLM} = r_1{}^{(0)} - \frac{b_{(0)}}{b_{(1)}} r_1{}^{(1)}. \tag{2.47}$$

The BLM procedure only determines the renormalization scale and thus it does not provide a full renormalization prescription even at NLO. Furthermore, it does not uniquely select the renormalization scale, as a N_f-independent rescaling will give identical expansions in $a(\mu^{BLM})$.

A major problem will now arise when we change scheme to one with different subtraction parameters and end up getting completely different results. This is precisely due to the fact that μ by itself does not specify the renormalization scheme, one needs in addition a $\tilde{\Lambda}_{RS}$ parameter. Some efforts to resolve this have been made [16] by the use of commensurate scale relations, it is proposed that all renormalization scheme dependence is removed, giving a completely unambiguous problem. It requires that the running coupling effects can be cleanly separated from the conformal part of the perturbative expansion of a generic physical quantity. This is currently unproven.

Despite its shortcomings, the idea at the base of BLM has inspired some interesting developments, namely the leading-b expansion in QCD which will be discussed in Section 4.5.

2.4.4 Effective Charge Approach

The Effective Charge (EC) approach, proposed by G. Grunberg [12, 13], is central to the renormalization scheme invariant resummation. Consider the generic dimensionless QCD observable $R(Q)$ with a formal perturbation series of the form Eq.(2.1). We shall refer to observables defined in this way as *effective charges*, which satisfy several important properties. The motivation behind this method is the recognition that for a dimensionless observable the energy scale dependence arises purely from the renormalised coupling. Thus by specifying an observable of the form Eq.(2.1) proportional

to the coupling and defined to all orders, we obtain an effective charge and as such we can apply the same renormalization group equation as satisfied by the coupling. This method amounts to choosing the renormalization scheme parameters such that r_1=r_2=...=r_k=0 rendering the renormalised coupling or effective charge, the actual observable. For the dimensionless QCD observable $R(Q)$ we can define the evolution equation for the Q-dependence of R

$$\frac{dR(Q)}{d\ln(Q)} = \xi(R). \tag{2.48}$$

$\frac{dR(Q)}{d\ln(Q)}$ and hence $\xi(R(Q))$ are in principle experimentally observable quantities. To make contact with QCD perturbation theory we note that Eq.(2.48) is the β-function equation in the effective charge scheme so we have

$$\frac{dR(Q)}{d\ln(Q)} = b\rho(R(Q)), \tag{2.49}$$

where

$$\rho(R(Q)) = -R^2(1 + \rho_1 R + \rho_2 R^2 + ... + \rho_n R^n + ...), \tag{2.50}$$

where the effective charge β-function coefficients ρ_n are Q-independent (but process-dependent) renormalization scheme invariant combinations of the r_i and $c_i (i \leq n)$. From Eq.(2.14) the effective charge scheme corresponds to τ=ρ_0 (ensuring r_1=0). To determine the remaining parameters, characterizing the EC renormalization scheme, one proceeds to follows. If we choose the barred scheme in Eq.(2.5) to be the EC scheme then

$$\bar{\beta}(\bar{a}) = b\rho(\bar{a}) = -b\bar{a}^2(1 + \rho_1\bar{a} + \rho_2\bar{a}^2 + ... + \rho_n\bar{a}^n + ...)\,, \tag{2.51}$$

with $\bar{a}$=R then Eq.(2.5) gives

$$\rho(R) = \beta(a(R))\frac{dR}{da}\,, \tag{2.52}$$

where $a(R)$ is the inverted perturbation series. By expanding both side of Eq.(2.52) as power series in R and equating coefficient we can obtain mathematical expressions for ρ_1, ..., ρ_n.

An easier way to get them is by considering appropriate expansions in Eq.(2.52) where

$$\begin{aligned}
\rho(R) &= -R^2(1+\rho_1 R+\rho_2 R^2+...)\\
R &= a+r_1a^2+r_2a^3+...\\
\beta(a) &= -ba^2(1+ca+c_2a^2+...)
\end{aligned} \tag{2.53}$$

and

$$\frac{dR}{da} = 1+2r_1a+3r_2a^2+... , \tag{2.54}$$

and substituting Eq.(2.53) and (2.54) in Eq.(2.52) we have

$$R^2(1+\rho_0 R+\rho_2 R^2+...) = a^2(1+ca+c_2a^2+...)(1+2r_1a+3r_2a^2+...), \tag{2.55}$$

substituting for R from (2.53) and equating coefficents of a^n in left and right-hand sides of Eq.(2.55) gives us

$$\begin{aligned}
\rho_1 &= c\\
\rho_2 &= c_2+r_2-cr_1-{r_1}^2\\
\rho_3 &= c_3+2r_3-4r_1r_2-4r_1\rho_2-c{r_1}^2+2{r_1}^3\\
&\vdots .
\end{aligned} \tag{2.56}$$

Rearranging Eq.(2.56) we exhibit the explicit τ, c_2, c_3, ... dependence of perturbation series coefficients

$$\begin{aligned}
r_1(\tau) &= \tau-\rho_0\\
r_2(\tau,c_2) &= (\tau-\rho_0)^2+c(\tau-\rho_0)+(\rho_2-c_2)\\
r_3(\tau,c_2,c_3) &= (\tau-\rho_0)^3+\frac{5}{2}c(\tau-\rho_0)^2\\
&+(3\rho_2-2c_2)(\tau-\rho_0)+\frac{1}{2}(\rho_3-c_3)\\
&\vdots\\
r_n(\tau,c_2,c_3,...) &= (\tau-\rho_0)^n+... .
\end{aligned} \tag{2.57}$$

The results for $r_n(\tau,c_2,c_3,...)$ is a polynomial of degree n in $(\tau-\rho_0)$ with coefficents involving ρ_n, ρ_{n-1}, ... and c, c_2,..., c_n such that $r_n(\rho_0,\rho_2,\rho_3,...,\rho_n)=0$. The ρ_2, ρ_3,

..., ρ_n, ... are process-dependent, renormalization scheme invariants which completely characterise the QCD observable R. They are independent of the energy scale Q but do depend on the number of active quark flavours and have special significance, while the renormalization scheme dependent r_k and c_k should be considered intermediate quantities to be eventually eliminated in favour of these renormalization scheme invariants.

The effective charge β-function $\rho(R(Q))$ is a key ingredient of the EC formalism and can be regarded as a physical observable to be reconstructed from the measured running of $R(Q)$, e.g. at LEP. As measured from data it will include a resummation of the (asymptotic) formal perturbation series together with non-perturbative $e^{\frac{-1}{bR}}$ terms which are invisible in perturbation theory and correspond to power corrections. First we need to integrate Eq.(2.48) to obtain $R(Q)$. We use the assumption of asymptotic freedom (AF) as the necessary boundary condiation. AF is equivalent to the statement that for any effective charge $R(Q)$

$$\lim_{Q\to\infty} R(Q) = 0, \tag{2.58}$$

this corresponds to the requirement that $\xi(R(Q))<0$ for $Q>Q_0$ with Q_0 some suitable low energy or equivalently $\rho(R(Q))>0$ for $Q>Q_0$. Integrating up Eq.(2.48) and imposing AF as a boundary condition we obtain

$$\ln(\frac{Q}{\Lambda_R}) = \int_0^{R(Q)} \frac{1}{\xi(x)} dx + (infinite \;\; constant)\,, \tag{2.59}$$

where Λ_R is a finite constant of integration which depends on the way the infinite constant is chosen. The infinite constant can be chosen to be

$$infinite \;\; constant = -\int_0^y \frac{dx}{\eta(x)}\,, \tag{2.60}$$

where $\eta(x)$ is any function which has the same $x \to 0$ behaviour as $\xi(x)$. This implies from Eq.(2.50) that

$$\eta(x) = -bx^2(1 + cx + K(x)), \tag{2.61}$$

where $K(x)$ is only constrained by the requirement that $\frac{K(x)}{x^2}$ is finite as $x \to 0$. Different choices of the upper limit of intergration and the function $K(x)$, can be absorbed

into the dimensionful constant Λ_R. Customary choices are $y = \infty$ and $K(x) = 0$ so

$$\eta(x) = -bx^2(1 + cx). \tag{2.62}$$

Inserting this choice for $\eta(x)$ and rearranging Eq.(2.59) we find

$$b\ln(\frac{Q}{\Lambda_R}) = \int_{R(Q)}^{\infty} \frac{dx}{x^2(1+cx)} + \int_0^{R(Q)} dx \left[\frac{-1}{\rho(x)} + \frac{1}{x^2(1+cx)}\right], \tag{2.63}$$

or alternatively

$$F(R(Q)) = bln(\frac{Q}{\Lambda_R}) - \int_0^{R(Q)} dx \left[\frac{-1}{\rho(x)} + \frac{1}{x^2(1+cx)}\right] \equiv b\ln\left(\frac{Q}{\Lambda_R}\right) - \Delta\rho_0. \tag{2.64}$$

where

$$F(x) = \frac{1}{x} + c\ln\left(\frac{cx}{1+cx}\right). \tag{2.65}$$

The integrand in $\Delta\rho_0$ is regular at x=0 and in arriving Eq.(2.64) we did not need to refer to perturbation theory except to assume the asymptotic $x \to 0 (Q \to \infty)$ behaviour $\eta(R(a)) = -bR^2(1 + cR)$ thus Eq.(2.64) holds beyond perturbation theory.

Rearranging Eq.(2.64) and assuming AF $(Q \to \infty, R(Q) \to 0)$, in which limit $\Delta\rho_0(Q) \to 0$, one finds that asymptotically for any effective charge $R(Q)$

$$\lim_{Q\to\infty} \mathcal{F}(R(Q)) = \Lambda_R\,. \tag{2.66}$$

Here Λ_R is an observable-dependent scaling constant with the dimension of energy and we have defined a universal QCD scaling function

$$\mathcal{F}(x) = e^{\frac{-F(x)}{b}} = e^{(\frac{-1}{bx})}\left(1 + \frac{1}{cx}\right)^{(\frac{c}{b})}. \tag{2.67}$$

The property (2.66) may be termed asymptotic scaling, and indeed it is exactly analogous to asymptotic scaling in lattice gauge theory which may be used to establish how close to the continuum limit of infinite inverse lattice spacing one is. Given sufficiently large values of Q this property can evidently serve as a test of QCD but since Λ_R is not universal it cannot usefully be applied at fixed values of Q. However if the NLO perturbative coefficent $r \equiv r_1^{(\overline{MS})}(\mu = Q)$ has been calculated, then Λ_R can be converted

into a universal scaling constant $\Lambda_{\overline{MS}}$ via the exact Celmaster-Gonsalves [18] relation (see Eq.(2.32))

$$\Lambda_R e^{(\frac{-r}{b})} = \Lambda_{\overline{MS}} \,. \tag{2.68}$$

Using the Eq.(2.68) and Eq.(2.66) one finds that, asymptotically, for any effective charge $R(Q)$

$$\lim_{Q\to\infty} \mathcal{F}(R(Q))e^{(\frac{-r}{b})} = \Lambda_{\overline{MS}} \,. \tag{2.69}$$

This property may be termed universal asymptotic scaling [17] and can be used to test QCD at fixed Q by looking at the scatter in $Q\mathcal{F}(R(Q))e^{(\frac{-r_1}{b})}$ for various observables. Using Eq.(2.14) we can write Eq.(2.64) as

$$F(R(Q)) = b\ln(\frac{Q}{\Lambda_{\overline{MS}}}) - r_1^{\overline{MS}} - \Delta\rho_0(Q) = \rho_0(Q) - \Delta\rho_0(Q) \tag{2.70}$$

as stressed by [19], Eq.(2.70) holds beyond perturbation theory for the measured observables $R(Q)$ and $\Delta\rho_0$ constructed from measured running of $R(Q)$ and its derivative with respect to Q, Eq.(2.48). No reference was made to perturbation theory except to assume AF that is we can write a $non-perturbative$ closed expression exactly relating the universal QCD dimensional transmutation parameter $\Lambda_{\overline{MS}}$ to physical obsrvables.

In the EC formalism above, the renormalised coupling a only ever appeared in intermediate steps. This of course, makes the question as to what is special about the effective charge scheme and why other choices of scale μ do not provide equally valid predictions for R. The key is to identify the way in which the Q-dependence of $R(Q)$ arises. In the construction above, it is built automatically by integration of Eq.(2.48) but how does it arise from the perturbation series in Eq.(2.1)?. For this purpose it will be more illuminating to consider an alternative formalism, proposed by C.J.Maxwell termed *Complete Renormalization Group Improvement.*

2.4.5 Complete Renormalization Group Improvement (CORGI)

This formalism treats the renormalization scale, μ as completely independent of the energy scale Q. This different perspective turns out to be entirely equivalent to the

EC formalism. So far all the solutions to the Renormalisation scheme dependent problem reviewed have been based on the truncating the perturbative series and choosing some scale μ=xQ. The dependence on the scale μ, which manifests itself via logarithms $\ln(\frac{\mu}{\Lambda_{\overline{MS}}})$ in $\overline{MS}$ scheme for instance, which are present in all perturbative orders, is thus replaced by a dependence on the energy scale Q. Here $\Lambda_{\overline{MS}}$ results as a combination of Eq.(2.68) and (2.4). Besides both having energy dimension, there is no reason why μ should be directly related to Q. In fact this formalism points out that the renormalization scale dependence of a dimensionless physical QCD observable, depending on a single energy scale Q, can be avoided provided that all ultraviolet logarithms which build the physical energy dependence on Q are resummed. It was stressed that standard renormalization group improvement , as customarily applied with a Q-dependent scale μ=xQ , omits an infinite subset of these logarithms. One should rather keep μ independent of Q, and then carefully resum to all orders the renormalization group predictable ultraviolet logarithms. In this way all μ-dependence cancels between the renormalized coupling and the logarithms of μ contained in the coefficents and the correct physical Q-dependence is built. At NLO the result is identical to the effective charge approach of Grunberg. The benefits of this formalism are not least a greater transparency in analysing the issues of scheme dependence, and the ability to implement higher order corrections through simple algebraic manipulation.

Now we turn to the RS-dependence of the perturbative coefficients r_i. This must be such as to cancel the RS-dependence of 'a' when the series is summed to all-orders. Using the self-consistency of perturbation theory [9] demands that the result of a $\mathrm{N^nLO}$ calculation (terms up to and including $r_n a^{n+1}$) in two *different* schemes should differ by $\mathrm{O}(a^{n+2})$, one can derive expressions for the partial derivatives of the perturbative coefficients with respect to the scheme parameters. Applied at next-to-NLO ($NNLO$) we find

$$\begin{aligned}\frac{dR^{(2)}}{dr_1} &= \frac{\partial a}{\partial r_1} + a^2 + 2r_1 a\frac{\partial a}{\partial r_1} + a^3\frac{\partial r_2}{\partial r_1} + 3r_2 a\frac{\partial a}{\partial r_1} + \ldots \\ \frac{dR^{(2)}}{dc_2} &= \frac{\partial a}{\partial c_2} + 2r_1 a\frac{\partial a}{\partial c_2} + a^3\frac{\partial r_2}{\partial c_2} + 3r_2 a\frac{\partial a}{\partial c_2} + \ldots .\end{aligned} \tag{2.71}$$

where the ellipsis denotes terms that start at $O(a^4)$. The dependence of the coupling a on scheme parameters c_2,...,c_n is obtained, using

$$\frac{\partial a}{\partial c_i} = -\beta(a) \int_0^a \frac{b x^{i+2}}{\beta^2(x)} dx \ , \tag{2.72}$$

which arises through the commutation relation

$$\frac{\partial a}{\partial c_i} \beta(a) - \beta(a) \frac{\partial a}{\partial c_i} = a^{i+2} \ , \tag{2.73}$$

(c.f [9]). Enforcing self-consistency requirements requires each lower order of the couplant to vanish independently. This leaves us with

$$\frac{\partial r_2}{\partial r_1} = 2r_1 + c, \quad \frac{\partial r_2}{\partial c_2} = -1, \quad \frac{\partial r_2}{\partial c_3} = 0, \ldots \ . \tag{2.74}$$

Since the only scheme parameters that r_2 will depend on, are r_1 and c_2, we can integrate Eq.(2.74) to obtain the most general form. Applying this formalism to each order of perturbation theory we arrive at an expression equivalent to Eqs.(2.57),

$$\begin{aligned} r_2(r_1, c_2) &= {r_1}^2 + c r_1 + X_2 - c_2 \\ r_3(r_1, c_2, c_3) &= {r_1}^3 + \frac{5}{2} c {r_1}^2 + (3X_2 - 2c_2) r_1 + X_3 - \frac{1}{2} c_3 \\ &\vdots \qquad \vdots \ . \end{aligned} \tag{2.75}$$

In general the structure is

$$r_n(r_1, c_2, \ldots, c_n) = \hat{r}_n(r_1, c_2, \ldots, c_{n-1}) + X_n - \frac{c_n}{(n-1)} \ , \tag{2.76}$$

where $\hat{r}_n$ is an n^{th}-order polynomial in r_1, and is renormalization group predictable from a complete $\mathrm{N}^{n-1}\mathrm{LO}$ calculation, i.e. r_2, r_3,..., r_{n-1} and c_2, c_3,..., c_{n-1} have been computed in some renormalization scheme and the X_n are Q-independent and renormalization scheme invariant constants of integration which are unknown unless a complete $\mathrm{N}^n\mathrm{LO}$ calculation has been performed. Given a NNLO calculation in the $\overline{MS}$ scheme with $\mu = Q$ for instance, one can determine the renormalization scheme invariant

$$X_2 = r_2^{\overline{MS}}(\mu = Q) - \left(r_1^{\overline{MS}}(\mu = Q)\right)^2 - c\, r_1^{\overline{MS}}(\mu = Q) + c_2^{\overline{MS}} \ , \tag{2.77}$$

Using Eqs.(2.77) we can now exhibit the explicit renormalization scheme dependence of the terms of Eq.(2.1),

$$\begin{aligned} R(Q) &= a + r_1 a^2 + (r_1^2 + c r_1 + X_2 - c_2)a^3 + (r_1^3 + \frac{5}{2} c r_1^2 + (3X_2 - 2c_2) r_1 \\ &\quad + X_3 - \frac{1}{2} c_3) a^4 + \ldots , \end{aligned} \tag{2.78}$$

where $a \equiv a(r_1, c_2, c_3, \ldots)$. Implementing the idea of CORGI means that at any given order of Feynman diagram calculation *all* known (renormalization group predictable) terms should be resummed to all-orders. Given a NLO calculation r_1 is known but $X_2, X_3, \ldots$ are unknown. Thus the complete subset of known terms in Eq.(2.78) at NLO is

$$a_0 \equiv a + r_1 a^2 + (r_1^2 + c r_1 - c_2)a^3 + (r_1^3 + \frac{5}{2} c r_1^2 - 2 c_2 r_1 - \frac{1}{2} c_3) a^4 + \ldots . \tag{2.79}$$

The sum of these terms, a_0 , can be simply determined by realising that this infinite subset has a renormalization scheme independent sum and that the X_2,X_3,... -dependent terms cannot affect this, and we know that the full sum of Eq.(2.78) is renormalization scheme invariant. Using the renormalization scheme independence we can set $r_1 = 0$, $c_2 = 0$, $c_3 = 0$, ..., in which case all terms but the first in Eq.(2.79) vanish and we obtain $a_0 = a(r_1 = 0, c_2 = 0, c_3 = 0, \ldots, c_n = 0, \ldots)$. So at NLO CORGI corresponds to working in a 't Hooft scheme with $c_2 = c_3 = \ldots = 0$ [20], and with $r_1 = 0$. From Eq.(2.14) $r_1 = 0$ corresponds to "Fastest Apparent Convergence"(FAC) or "Effective Charge" (EC) scale. From the generalized from of Eq.(2.8) [27]

$$\frac{1}{a} + c \ln\left(\frac{ca}{1+ca}\right) = \tau - \int_0^\infty dx \left[-\frac{1}{B(x)} + \frac{1}{x^2(1+cx)} \right] , \tag{2.80}$$

where $B(x) \equiv x^2(1 + cx + c_2 x^2 + c_3 x^3 + \ldots)$ and a_0 satisfies

$$\frac{1}{a_0} + c \ln\left(\frac{ca_0}{1+ca_0}\right) = b \ln\left(\frac{Q}{\Lambda_R}\right) . \tag{2.81}$$

In fact the solution of this transcendental equation can be written in closed form in terms of the Lambert W-function [21,22], defined implicitly by $W(z)\exp(W(z)) = z$,

$$\begin{aligned} a_0 &= -\frac{1}{c[1 + W(z(Q))]} , \\ z(Q) &\equiv -\frac{1}{e}\left(\frac{Q}{\Lambda_R}\right)^{-b/c} . \end{aligned} \tag{2.82}$$

To see how Eq.(2.82) satisfies the Lambert-W function, by exponentiating Eq.(2.81) we can write it as

$$e^{(\frac{1}{a_0})}\Big(\frac{ca_0}{1+ca_0}\Big)^c = \Big(\frac{Q}{\Lambda_R}\Big)^b , \tag{2.83}$$

or

$$e^{(\frac{1}{a_0})}\Big(1+\frac{1}{ca_0}\Big)^{-c} = \Big(\frac{Q}{\Lambda_R}\Big)^b . \tag{2.84}$$

Following simplification we have

$$e^{(\frac{-1}{ca_0})}\Big(1+\frac{1}{ca_0}\Big) = \Big(\frac{Q}{\Lambda_R}\Big)^{\frac{-b}{c}} , \tag{2.85}$$

and finally

$$e^{-(1+\frac{1}{ca_0})}\Big(1+\frac{1}{ca_0}\Big) = \frac{1}{e}\Big(\frac{Q}{\Lambda_R}\Big)^{\frac{-b}{c}} . \tag{2.86}$$

If we assume $-(1+\frac{1}{ca_0}) = W(z)$ then Eq.(2.86) is equivalent to $W(z)\exp(W(z)) = -\frac{1}{e}\Big(\frac{Q}{\Lambda_R}\Big)^{\frac{-b}{c}}$ where $z = -\frac{1}{e}\Big(\frac{Q}{\Lambda_R}\Big)^{-b/c}$.

If a NNLO calculation has been completed , then X_2 can be determined (as in Eq.(2.77)), and a further infinite subset of terms are known and can be resummed to all-orders,

$$X_2 {a_0}^3 = X_2 a^3 + 3X_2 r_1 a^4 + \dots . \tag{2.87}$$

The renormalization scheme independence of the sum and the multinomial structure of the coefficients again leads to a resummed result involving a_0. By recognising that the combination of terms proportional to X_n are equivalent to a_0^{n+1} results in

$$R(Q) = a_0 + X_2 a_0^3 + X_3 a_0^4 + \dots + X_n a_0^{n+1} + \dots , \tag{2.88}$$

which is simply the perturbation series in the renormalization scheme with $r_1 = c_2 = c_3 = \dots = c_n = \dots = 0$.

At the NLO result we see that in evaluating a_0, we are really resumming all the r_1-dependence to all orders. This effectively resums all the $\ln(\frac{\mu}{\Lambda_{RS}})$ terms as has traditionally been advocated in renormalization group improvement with the additional

$\ln(\frac{Q}{\Lambda_R})$ that necessarily appear as previously indicated. This is highlighted by rewriting Eq.(2.14) as

$$r_1 = b\left(ln(\frac{\mu}{\tilde{\Lambda}_{RS}}) - ln(\frac{Q}{\Lambda_R})\right) . \tag{2.89}$$

The first μ-dependent logarithm depends on the renormalization scheme, whereas the second Q-dependent UV logarithm will generate the physical Q-dependence and is renormalization scheme invariant. If one makes the simplification that $c = 0$ and sets $c_2 = c_3 = ... = 0$, then the coupling is given by

$$a(\mu) = \frac{1}{b\ln(\frac{\mu}{\tilde{\Lambda}_{RS}})} . \tag{2.90}$$

The sum to all-orders of the RG-predictable terms from Eqs.(2.78), given a NLO calculation of r_1, simplifies to a geometric progression,

$$R = a + r_1 a^2 + {r_1}^2 a^3 + \ldots + {r_1}^n a^{n+1} + \ldots . \tag{2.91}$$

The idea of complete RG-improvement is that dimensionful renormalization scales, in this case μ, should be held strictly independent of the physical energy scale Q on which $\mathcal{R}(Q)$ depends. In this way the Q-dependence is built entirely by the "physical" UV logarithms $b\ln(Q/\Lambda_{\mathcal{R}})$ contained in r_1, and the convention-dependent logarithms of μ cancel between $a(\mu)$ and $r_1(\mu)$, when the all-orders sum in Eq.(2.91) is evaluated. The conventional fixed-order NLO truncation $\mathcal{R} = a(\mu) + r_1(\mu)a(\mu)^2$, only makes sense if $\mu = xQ$, but then the resulting Q-dependence involves the arbitrary parameter x. In contrast using Eqs.(2.89),(2.90) and summing the geometric progression in Eq.(2.91) gives,

$$R(Q) \approx a(\mu) / \left[1 - \left(b\ln\frac{\mu}{\tilde{\Lambda}_{RS}} - b\ln\frac{Q}{\Lambda_R}\right) a(\mu)\right] = 1/b\ln(Q/\Lambda_R) , \tag{2.92}$$

in which the unphysical μ-dependence has cancelled between $a(\mu)$ and the μ-dependent logarithms contained in $r_1(\mu)$. One has therefore directly traded unphysical μ-dependence for the physical Q-dependence.

The extension of this argument to processes involving factorization of operator matrix elements and coefficient functions, where a factorization scale M arises in addition to the renormalization scale μ, will be discussed in the next chapter.

Chapter 3

Factorization and Renormalization Scale dependence

3.1 Extension of CORGI to Two Scales

In the last chapter it was stressed that standard RG-improvement, as customarily applied with a Q-dependent scale $\mu = xQ$, omits an infinite subset of RG predictable UV logarithms. One should rather keep μ independent of Q, and then carefully resum to all-orders the RG-predictable ultraviolet logarithms. In this way all μ-dependence cancels between the renormalized coupling and the logarithms of μ contained in the coefficients, and the correct physical Q-dependence is built. At next-to-leading order (NLO) the result is identical to the Effective Charge approach of Grunberg [12,13]. We wish to extend this argument to processes involving factorization of operator matrix elements and coefficient functions, where a factorization scale M arises in addition to the renormalization scale μ. We shall use the prototypical factorization problem of moments of leptoproduction structure functions as a specific example. We shall identify the logarithms of μ, M, and Q which occur, and will show explicitly that on resumming all the ultraviolet logarithms the μ and M dependence disappears. We begin Section 2 by giving some basic definitions for the moments of structure functions.

3.2 Structure Function moments

In the prototypical factorization problem of deep inelastic leptoproduction the n^{th} moment of a non-singlet structure function $F(x)$,

$$\mathcal{M}_n(Q) = \int_0^1 x^{n-2} F(x)\, dx \,, \tag{3.1}$$

can be factorized in the form

$$\mathcal{M}_n(Q) = < \mathcal{O}_n(M) > \mathcal{C}_n(Q, a(\mu), \mu, M) \,. \tag{3.2}$$

Here M is an arbitrary factorization scale and $a(\mu)$ is the RG-improved coupling $\alpha_s(\mu)/\pi$ defined at a renormalization scale μ. The operator matrix element $< \mathcal{O}_n(M) >$ has an M-dependence given by its anomalous dimension,

$$\frac{M}{< \mathcal{O} >} \frac{\partial < \mathcal{O} >}{\partial M} = \gamma_{\mathcal{O}}(a) = -da - d_1 a^2 - d_2 a^3 - d_3 a^4 + \ldots \tag{3.3}$$

For simplicity we shall from now on suppress the n-dependence of terms in equations, as we have done in Eq.(3.3). For a given moment d is independent of the factorization convention, whereas the higher d_i, ($i \geq 1$) depend on it. Recall that in Eq.(3.3) the coupling a is governed by the β-function equation

$$M \frac{\partial a}{\partial M} = \beta(a) = -ba^2(1 + ca + c_2 a^2 + c_3 a^3 + \ldots) \,. \tag{3.4}$$

Here b and c are the first two coefficients which are universal (c.f. (1.59)), whereas the subsequent coefficients $c_2, c_3, \ldots$ are scheme-dependent. Equation (3.3) can be integrated to [25, 26]

$$< \mathcal{O}(M) > = A \exp[\int_0^a \frac{\gamma(x)}{\beta(x)}\, dx - \int_0^\infty \frac{\gamma^{(1)}(x)}{\beta^{(2)}(x)}\, dx] \,, \tag{3.5}$$

where $\gamma^{(1)}$ and $\beta^{(2)}$ denote these functions truncated at one and two terms, respectively. The factor A is scheme-independent [26] and can be fitted to experimental data. The second integral in Eq.(3.5) is an infinite constant of integration. In Eq.(3.2) $\mathcal{C}(Q, a(\mu), \mu, M)$ is the coefficient function and has the perturbation series

$$\mathcal{C}(Q, \tilde{a}, \mu, M) = 1 + r_1 \tilde{a} + r_2 \tilde{a}^2 + r_3 \tilde{a}^3 + \ldots \,. \tag{3.6}$$

We shall use $\tilde{a}$ to stand for $a(\mu)$ and a for $a(M)$. After combining the integrals in Eq.(3.5) one obtains

$$\mathcal{M} = A\Big(\frac{ca}{1+ca}\Big)^{d/b} \exp(\mathcal{I}(a))\,(1 + r_1\tilde{a} + r_2\tilde{a}^2 + r_3\tilde{a}^3 + \ldots)\,. \tag{3.7}$$

where $\mathcal{I}(a)$ is the finite integral

$$\mathcal{I}(a) = \int_0^a dx\, \frac{d_1 + (d_1c + d_2 - dc_2)x + (d_3 + cd_2 - c_3d)x^2 + \ldots}{b(1+cx)(1+cx+c_2x^2+c_3x^3+\ldots)}\,, \tag{3.8}$$

which can be readily evaluated numerically.Recall that the coupling $a(\tau)$ itself, where $\tau \equiv b\ln(\mu/\tilde{\Lambda})$. is obtained as the solution of the transcendental equation (c.f. Eq.(2.81))

$$\frac{1}{a} + c\ln\frac{ca}{1+ca} = \tau - \int_0^a dx\left[-\frac{1}{B(x)} + \frac{1}{x^2(1+cx)}\right]\,. \tag{3.9}$$

By expanding (3.8) in terms of x and doing the integration and combining the obtained result with the expanded form of $\left(\frac{ca}{1+ca}\right)^{d/b}$ we arrive at the following result

$$\mathcal{M} = (ca)^{(\frac{d}{b})}(1 + pa + qa^2 + ra^3 + ...)(1 + r_1\tilde{a} + r_2\tilde{a}^2 + r_3\tilde{a}^3 + ...) \tag{3.10}$$

where

$$p = -\frac{dc}{b} + \frac{d_1}{b} \tag{3.11}$$

$$q = \frac{1}{2}p^2 + \frac{1}{2}\frac{d_2}{b} + \frac{1}{2}\frac{d\,(c^2 - c_2)}{b} - \frac{1}{2}\frac{d_1\,c}{b} \tag{3.12}$$

$$\begin{aligned} r \;=\; & \frac{-1}{3}p^3 + pq + \frac{1}{3}\frac{d_3}{b} - \frac{1}{3}\frac{d_2c}{b} - \frac{1}{3}\frac{d_1c_2}{b} + \frac{1}{3}\frac{d_1c^2}{b} \\ & - \frac{1}{3}\frac{dc^3}{b} - \frac{1}{3}\frac{dc_3}{b} + \frac{2}{3}\frac{dcc_2}{b} \end{aligned} \tag{3.13}$$

In Appendix A we tried to get a closed analytical form for moments $\mathcal{M}$ that is valid up to a^4 and in some senses has some advantages with repect to the expansion form (3.10).

3.3 RS and FS dependence of the coefficients

We first wish to parametrize the dependence of the r_n in the coefficient function on the renormalization scheme (RS) and factorization scheme (FS). For the single scale case we saw in Chapter 2 that the RS could be parametrised by r_1 and the non-universal β-function coefficients c_2, c_3,... .

As we shall see the generalization to the moment problem is a dependence $r_n(\mu, M, c_2, \ldots, c_n, d_1, d_2, \ldots, d_n)$ where the c_i label the RS and the d_i the FS. As before M, μ can be traded, in this case for $r_1(M)$ and $\tilde{r}_1 \equiv r_1(M = \mu)$. There will be analogous factorization and renormalization scheme (FRS) invariants,X_n , which represent the RG-unpredictable parts of r_n. Expressions for the dependence of the coefficients on FRS parameters have been derived before in Refs. [25, 26, 28], but there were some errors in Ref. [25], in particular the dependence of r_2 on c_2 was not recognized [26].

Partially differentiating Eq.(3.10) or using the closed form of $\mathcal{M}$ where the above approximant depends on seven unphysical variables μ, M, d_1, d_2, d_3, c_2, and c_3 yields (see appendix B)

$$\begin{aligned} \mu\frac{\partial \mathcal{M}}{\partial \mu} &= \frac{\mathcal{M}}{1 + r_1\tilde{a} + r_2\tilde{a}^2 + r_3\tilde{a}^3}[-ba^2(1 + ca + c_2a^2 + c_3a^3) \times \\ &\quad (r_1 + 2r_2\tilde{a}^1 + 3r_3\tilde{a}^2) + \mu\frac{\partial r_1}{\partial \mu}\tilde{a} + \mu\frac{\partial r_2}{\partial \mu}\tilde{a}^2 + \mu\frac{\partial r_3}{\partial \mu}\tilde{a}^3] , \end{aligned} \tag{3.14}$$

$$M\frac{\partial \mathcal{M}}{\partial M} = \mathcal{M}[-da - d_1a^2 - d_2a^3 - d_3a^4 + \frac{M\frac{\partial r_1}{\partial M}\tilde{a} + M\frac{\partial r_2}{\partial M}\tilde{a}^2 + M\frac{\partial r_3}{\partial M}\tilde{a}^3}{1 + r_1\tilde{a} + r_2\tilde{a}^2 + r_3\tilde{a}^3}] , \tag{3.15}$$

$$\begin{aligned} \frac{\partial \mathcal{M}}{\partial d_1} &= \mathcal{M}[log(1 + ca)(\frac{-1}{bc} - \frac{2c_2}{bc^3} + \frac{3c_3}{bc^4}) + c_2(\frac{ca^2 + 2a}{bc^2(1 + ca)} \\ &\quad -c_3\frac{(-c^3a^3 + 3c^2a^2 + 6ca)}{2c^4(1 + ca)b} + \frac{\frac{\partial r_1}{\partial d_1}\tilde{a} + \frac{\partial r_2}{\partial d_1}\tilde{a}^2 + \frac{\partial r_3}{\partial d_1}\tilde{a}^3}{1 + r_1\tilde{a} + r_2\tilde{a}^2 + r_3\tilde{a}^3}] , \end{aligned} \tag{3.16}$$

$$\frac{\partial \mathcal{M}}{\partial d_2} = \mathcal{M}[log(1+ca)(\frac{-1}{bc^2}+\frac{3c_2}{bc^4}-\frac{4c_3}{bc^5})-\frac{a}{bc}-c_2\frac{(-c^3a^3+3c^2a^2+6ca)}{2c^4(1+ca)b} + c_3\frac{(12a+6ca^2-2c^2a^3+c^3a^4)}{c^4(1+ca)b}+\frac{\frac{\partial r_1}{\partial d_2}\tilde{a}+\frac{\partial r_2}{\partial d_2}\tilde{a}^2+\frac{\partial r_3}{\partial d_2}\tilde{a}^3}{1+r_1\tilde{a}+r_2\tilde{a}^2+r_3\tilde{a}^3}] . \quad (3.17)$$

$$\frac{\partial \mathcal{M}}{\partial d_3} = \mathcal{M}[log(1+ca)(\frac{-1}{bc^3}-\frac{4c_2}{bc^5}+\frac{5c_3}{bc^6})-a(\frac{c^2a^2-2-ca}{bc(1+ca)}) +c_2\frac{(c^3a^4-2c^2a^3+6ca^2+12a)}{3c^4(1+ca)b} -c_3\frac{(60a+30ca^2-10c^2a^3+5c^3a^4-3c^4a^5)}{12bc^5(1+ca)} +\frac{\frac{\partial r_1}{\partial d_3}\tilde{a}+\frac{\partial r_2}{\partial d_3}\tilde{a}^2+\frac{\partial r_3}{\partial d_3}\tilde{a}^3}{1+r_1\tilde{a}+r_2\tilde{a}^2+r_3\tilde{a}^3}] . \quad (3.18)$$

$$\frac{\partial \mathcal{M}}{\partial c_2} = \mathcal{M}[log(1+ca)(\frac{-d}{bc^2}+\frac{2d_1}{bc^3}-\frac{3d_2}{bc^4}+\frac{4d_3}{c^5}) +\frac{ad}{bc(1+ca)}-d_1(\frac{2a+ca^2}{bc^2(1+ca)})+d_2\frac{(-c^2a^3+3ca^2+60a)}{2c^3(1+ca)b} {}_{d_3}\frac{(12a+6ca^2-2c^2a^3+c^3a^4)}{12bc^4(1+ca)}+\frac{\frac{\partial r_1}{\partial c_2}\tilde{a}+\frac{\partial r_2}{\partial c_2}\tilde{a}^2+\frac{\partial r_3}{\partial c_2}\tilde{a}^3}{1+r_1\tilde{a}+r_2\tilde{a}^2+r_3\tilde{a}^3}] . \quad (3.19)$$

$$\frac{\partial \mathcal{M}}{\partial c_3} = \mathcal{M}[log(1+ca)(\frac{2d}{bc^3}-\frac{3d_1}{bc^4}+\frac{4d_2}{bc^5}-\frac{5d_3}{bc^6})-d\frac{(ca^2+2a)}{bc^2(1+ca)} +d_1\frac{(-c^3a^3+3c^2a^2-6ca)}{2c^4(1+ca)b}-d_2\frac{(12a+6ca^2-2c^2a^3+c^3a^4)}{3bc^4(1+ca)} +d_3\frac{(60a+6ca^2-10c^2a^3+5c^3a^4-3c^4a^5)}{12bc^5(1+ca)} +\frac{\frac{\partial r_1}{\partial c_3}\tilde{a}+\frac{\partial r_2}{\partial c_3}\tilde{a}^2+\frac{\partial r_3}{\partial c_3}\tilde{a}^3}{1+r_1\tilde{a}+r_2\tilde{a}^2+r_3\tilde{a}^3}] . \quad (3.20)$$

In obtaining the above derivatives with respect to c_2 and c_3 we used Eq.(2.73) and the analogous one for $\tilde{a}$ that gives the c_2 and c_3 dependence of a and $\tilde{a}$ (c.f. [9]).

Demanding for consistency that these partial derivatives are $O(a^4)$, so that the

coefficients of a, a^2 and a^3 vanish, one obtains analogous to Eqs.(2.75),

$$\begin{aligned}
\mu\frac{\partial r_1}{\partial \mu} &= 0, \quad \mu\frac{\partial r_2}{\partial \mu} = r_1 b, \quad \mu\frac{\partial r_3}{\partial \mu} = 2r_2 b + r_1 bc\,, \\
M\frac{\partial r_1}{\partial M} &= d, \quad M\frac{\partial r_2}{\partial M} = d_1 + dr_1 - dL\,, \\
M\frac{\partial r_3}{\partial M} &= d_2 + d_1 r_1 + dr_2 - dr_1 L - 2d_1 L - dL^2, \\
\frac{\partial r_1}{\partial d_1} &= -\frac{1}{b}, \quad \frac{\partial r_2}{\partial d_1} = \frac{c}{2b} - \frac{L}{b} - \frac{r_1}{b}, \\
\frac{\partial r_3}{\partial d_1} &= \frac{cr_1}{2b} - \frac{c^2}{3b} + \frac{(c - r_1)}{b}L - \frac{r_2}{b} + \frac{c_2}{3b} - \frac{L^2}{b}, \\
\frac{\partial r_1}{\partial d_2} &= 0, \quad \frac{\partial r_2}{\partial d_2} = -\frac{1}{2b}, \quad \frac{\partial r_3}{\partial d_2} = \frac{c}{3b} - \frac{L}{b} - \frac{r_1}{2b}\,, \\
\frac{\partial r_1}{\partial d_3} &= 0, \quad \frac{\partial r_2}{\partial d_3} = 0, \quad \frac{\partial r_3}{\partial d_3} = -\frac{1}{3b}\,, \\
\frac{\partial r_1}{\partial c_2} &= 0, \quad \frac{\partial r_2}{\partial c_2} = \frac{3d}{2b}, \quad \frac{\partial r_3}{\partial c_2} = \frac{4d_1}{3b} + 3\frac{dL}{b} + 3\frac{dr_1}{2b} - r_1 - 5\frac{cd}{3b}\,, \\
\frac{\partial r_1}{\partial c_3} &= 0, \quad \frac{\partial r_2}{\partial c_3} = 0, \quad \frac{\partial r_3}{\partial c_3} = \frac{5d}{6b}\,.
\end{aligned} \tag{3.21}$$

As we expected, there is not any dependence on c_2, c_3, d_2, d_3 in r_1 and c_3, d_3 in r_2 (see Appendix B).

Here we have defined for convenience $L{\equiv}b\ln(M/\mu)$. Consistently integrating the partial derivatives of r_1 yields

$$r_1 = \frac{d}{b}\tau_M - \frac{d_1}{b} - X_1(Q)\,, \tag{3.22}$$

where $\tau_M{\equiv}b\ln(M/\tilde{\Lambda})$ and $X_1(Q)$ is an FRS-invariant, analogous to $\rho_0(Q)$ for the single scale problem defined in Eq.(2.14). Exactly analogous to $\Lambda_{\mathcal{R}}$, for the moment problem one can define an FRS-invariant $\Lambda_{\mathcal{M}}$ so that

$$\frac{d}{b}\tau_M - \frac{d_1}{b} - r_1 = X_1(Q){\equiv}d\ln\left(\frac{Q}{\Lambda_{\mathcal{M}}}\right)\,. \tag{3.23}$$

Consistently integrating the remaining partial derivatives and using Eq.(3.22) to recast the M and μ dependence in terms of r_1 and $\tilde{r}_1$ ($\partial log M = \frac{\partial r_1}{d}$ and $\partial log\mu = \frac{\partial \tilde{r}_1}{d}$), one obtains the explicit dependence of r_2 and r_3 on the FRS parameters $r_1, \tilde{r}_1, d_1, d_2, d_3, c_2, c_3$:

$$r_2 = (\frac{1}{2} - \frac{b}{2d}){r_1}^2 + \frac{b}{d}r_1\tilde{r}_1 + \frac{cd_1}{2b} - \frac{d_2}{2b} - \frac{dc_2}{2b} + X_2$$

$$
\begin{aligned}
r_3 = &(\frac{b^2}{d^2} - \frac{3b}{2d} + \frac{1}{2})\frac{{r_1}^3}{3} + (-\frac{b^2}{d^2} + \frac{b}{d}){r_1}^2\tilde{r}_1 + (\frac{bc}{d} + \frac{2bd_1}{d^2})r_1\tilde{r}_1 \\
&+(-\frac{bc}{2d} - \frac{bd_1}{d^2} + \frac{d_1}{d}){r_1}^2 + (-\frac{dc_2}{2b} + \frac{cd_1}{2b} + X_2 + \frac{{d_1}^2}{2db} + \frac{d_2}{d} - \frac{d_2}{2b} - c_2)r_1 \\
&+(\frac{{d_1}^2}{d^2} - \frac{d_2}{d} + \frac{cd_1}{d} + \frac{2bX_2}{d})\tilde{r}_1 + \frac{b^2}{d^2}r_1\tilde{r}_1^2 + (-\frac{d_1c^2}{3b} + \frac{2d_1X_2}{d} \\
&+\frac{{d_1}^3}{3bd^2} + \frac{dcc_2}{3b} + \frac{cd_1^2}{2db} + \frac{d_3}{3d} - \frac{dc_3}{6b} - \frac{2d_1c_2}{3b} + \frac{d_2c}{3b} + X_3) \\
&\vdots \quad \vdots ,
\end{aligned}
\tag{3.24}
$$

analogous to Eqs.(2.76) in the single scale case. Notice that we could equally use r_1 and L as parameters instead of r_1 and $\tilde{r}_1$, since $L = (\frac{b}{d})(r_1 - \tilde{r}_1)$. As in the single scale case there are constants of integration X_n representing the RG-unpredictable part of r_n. They are Q-independent and FRS-invariant.

Recall that in the single scale case parametrizing the RS-dependence using $r_1, c_2, c_3, \ldots$ means that given a complete N^{n}LO calculation $X_2, X_3, \ldots, X_n$ will be known. Using Eqs.(2.76) to sum to all-orders the RG-predictable terms, i.e. those *not* involving $X_{n+1}, X_{n+2}, \ldots$, with coupling $a(r_1, c_2, c_3, \ldots)$ is equivalent to N^{n}LO perturbation theory in the scheme with $r_1 = c_2 = c_3 = \ldots = 0$, and yields the sum

$$
\mathcal{R}^{(n)} = a_0 + X_2 a_0^3 + X_3 a_0^4 + \ldots + X_n a_0^{n+1} , \tag{3.25}
$$

where $a_0 \equiv a(0, 0, 0, \ldots)$ is the coupling in this scheme and satisfies Eq.(2.83) in terms of the Lambert-W function.

In the moment problem by an exactly similar argument, with the chosen parametrization of FRS, given a complete N^{n}LO calculation (i.e. a calculation of $r_1, r_2, \ldots, r_n$ and the $d_1, d_2, \ldots, d_n$ and $c_2, c_3, \ldots, c_n$ in some FRS) the invariants $X_2, X_3, \ldots, X_n$ will be known. Using Eqs.(3.24) to sum to all-orders the RG-predictable terms not involving $X_{n+1}, X_{n+2}, \ldots$, will be equivalent to working with an FRS in which all the FRS parameters are set to zero. $\tilde{r}_1 = 0$ means that $\mu = M$. Setting $r_1 = 0$, $d_1 = 0$ in Eq.(3.24) yields $\tau_M = b\ln(Q/\Lambda_{\mathcal{M}})$, so that $a = \tilde{a} = a_0$, given by Eq.(2.83) with $\Lambda_{\mathcal{R}}$ replaced by $\Lambda_{\mathcal{M}}$. Further, with $c_i = d_i = 0$ the integral $\mathcal{I}(a)$ in Eq.(3.8) vanishes, so that finally

the sum of all RG-predictable terms for the moment problem at N^{n}LO will be

$$\mathcal{M} = A\Big(\frac{ca_0}{1+ca_0}\Big)^{d/b}(1 + X_2a_0^2 + X_3a_0^3 + \ldots + X_na_0^n)\,, \tag{3.26}$$

with an extremely similar structure to the single scale case in Eq.(3.25). Substituting for a_0 in terms of the Lambert W-function using Eq.(2.83) we then obtain

$$\begin{aligned}\mathcal{M} &= A[-W(z(Q))]^{b/d}(1 + X_2a_0^2 + \ldots)\\ z(Q) &\equiv -\frac{1}{e}\Big(\frac{Q}{\Lambda_{\mathcal{M}}}\Big)^{-b/c}\,.\end{aligned} \tag{3.27}$$

So that moments of structure functions have a Q-dependence naturally involving a power of the Lambert W-function.

The result of resumming all RG-predictable terms depends on the chosen parametrization of RS [29]. By simply translating the parameters to a new set $\breve{r}_1 = r_1 - \bar{r}_1$, $\breve{c}_2 = c_2 - \bar{c}_2, \ldots$ etc., where the barred quantities are constants, one finds corresponding new constants of integration $\breve{X}_n$. The result of resumming all RG-predictable terms with this new parametrization then corresponds to standard fixed-order perturbation theory in the RS with $r_1 = \bar{r}_1$, $c_2 = \bar{c}_2, \ldots$, or equivalently with $\breve{r}_1 = \breve{c}_2 = \breve{c}_3 = \ldots = 0$. The key point is that r_1 has a special status since it contains the ultraviolet (UV) logarithms which build the physical Q-dependence of $\mathcal{R}(Q)$ (c.f. Eq.(2.90)). Standard RG-improvement corresponds to shifting the parameter r_1, in which case the resulting constants of integration $\breve{X}_n$ contain physical UV logarithms which are not all resummed. Thus r_1 should be used as the parameter. An exactly similar statement holds for r_1 and $\bar{r}_1$ in the moment problem. We shall identify the UV logarithms and show how their complete resummation builds the correct physical Q-dependence in the next section.

We shall refer to the expansions in Eqs.(3.26) and (3.27) as Complete RG-improved (CORGI) results. Again for the moment problem the parameters implicitly containing the UV logarithms do have a special status, whilst the remaining dimensionless parameters c_i and d_i can be reparametrized as one pleases. As an example, in the Effective

Charge approach of Grunberg [12, 13] one chooses $\bar{c}_2, \bar{c}_3, \ldots, \bar{c}_n$ so that $\breve{X}_2, \breve{X}_3, \ldots, \breve{X}_n$ are all zero at N^{n}LO, corresponding to $r_1 = r_2 = \ldots = r_n = 0$, and this is *a priori* equally reasonable. In the moment problem one can correspondingly choose the $\bar{c}_i$ and $\bar{d}_i$ so that at N^{n}LO the $\breve{X}_i$ all vanish and $r_1 = r_2 = \ldots = r_n = 0$. If one further demands that the integral $\mathcal{I}(a)$ in Eq.(3.8) vanishes order-by-order in a a unique FRS is selected in which moments have the form

$$\mathcal{M} = A\left(\frac{c\hat{\mathcal{R}}}{1 + c\hat{\mathcal{R}}}\right)^{d/b} , \tag{3.28}$$

where $\hat{\mathcal{R}}$ is an effective charge which has a perturbation series of the form,

$$\hat{\mathcal{R}} = a + \hat{r}_1 a^2 + \hat{r}_2 a^3 + \ldots + \hat{r}_n a^{n+1} + \ldots . \tag{3.29}$$

This is similar to Grunberg's proposal [13] to associate an effective charge with $\mathcal{M}$ so that $\mathcal{M} = A(c\hat{\mathcal{R}})^{d/b}$. The $\hat{r}_i$ are built from the c_i, d_i, M and μ, and are RS-dependent, but FS-independent. Effectively $\hat{\mathcal{R}}$ can be RG-improved as in the single scale case. We have for instance

$$\hat{r}_1 = b\ln(\mu/\tilde{\Lambda}) - b\ln(M/\tilde{\Lambda}) - \frac{b}{d}r_1 + d_1/d = \tau - \frac{b}{d}X_1(Q) , \tag{3.30}$$

where we have used Eq.(3.23) . Comparing with Eq.(2.14) we see that treating $\hat{\mathcal{R}}$ as a single scale problem we have $\rho_0(Q) = \frac{b}{d}X_1(Q)$. This further implies that $\Lambda_{\hat{\mathcal{R}}} = \Lambda_{\mathcal{M}}$ and so the corresponding CORGI couplings are identical. The CORGI expansion for $\hat{\mathcal{R}}$ will be of the form

$$\hat{\mathcal{R}} = a_0 + \hat{X}_2 a_0^3 + \hat{X}_3 a_0^4 + \ldots . \tag{3.31}$$

Inserting this result in Eq.(3.28) and re-expanding in a_0 will reproduce the CORGI expansion in Eq.(3.26).

3.4 CORGI and Structure Function moments

In the moment problem the analogous UV logarithm is $d\ln(Q/\Lambda_{\mathcal{M}})$ introduced in Eq.(3.23), and analogous to Eq.(2.90) we will have

$$r_1 = d\left(\ln\frac{M}{\tilde{\Lambda}} - \ln\frac{Q}{\Lambda_{\mathcal{M}}}\right) - \frac{d_1}{b} . \tag{3.32}$$

Given a NLO calculation of r_1 we wish to see how the physical Q-dependence of $\mathcal{M}(Q)$ arises on resumming to all-orders the UV logarithms contained in the RG-predictable terms from Eqs.(3.24). If we make similar approximations, so that $c = 0$ and the d_i and c_i are set to zero, then

$$\mathcal{M} = A(ca(M))^{d/b}(1 + r_1 a(\mu) + r_2 a(\mu)^2 + \ldots) . \tag{3.33}$$

We retain the overall factor of $c^{d/b}$. The task is then to show that on resumming the RG-predictable terms in the coefficient function to all-orders the $\ln(M/\tilde{\Lambda})$ and $\ln(\mu/\tilde{\Lambda})$ contained in r_1 and $\tilde{r}_1$ cancel with those in the couplings $a(M)$ and $a(\mu)$ to yield the physical Q-dependence

$$\mathcal{M}(Q) \approx Ac^{d/b}(1/b\ln(Q/\Lambda_{\mathcal{M}}))^{d/b}(1 + O(1/\ln(Q/\Lambda_{\mathcal{M}}))^2) . \tag{3.34}$$

Again, the complete RG-improvement summing over all UV logarithms is forced on one if μ and M are held independent of Q.

The algebraic structure of the resummation of RG-predictable terms for the moment problem is considerably more complicated than the geometric progression of Eq.(2.92) encountered in the single scale case. With the simplifications $c = 0$, $c_i = 0$, $d_i = 0$ the first two RG-predictable coefficients from Eqs(3.25) are

$$r_2 = (\frac{1}{2} - \frac{b}{2d}){r_1}^2 + \frac{b}{d} r_1 \tilde{r_1} \tag{3.35}$$

$$r_3 = (\frac{b^2}{d^2} - \frac{3b}{2d} + \frac{1}{2})\frac{{r_1}^3}{3} + (\frac{-b^2}{d^2} + \frac{b}{d}){r_1}^2 \tilde{r_1} + \frac{b^2}{d^2} r_1 {\tilde{r_1}}^2 . \tag{3.36}$$

Suitably generalizing the partial derivatives in Eqs.(3.21) one can arrive at a general form for the RG-predictable terms. For instance by considering the derivatives of r_4,r_5,... with respect to r_1 and $\tilde{r_1}$ we will have

$$\begin{aligned} r_4 = {} & (\frac{b}{2d} - \frac{3b^2}{2d^2} + \frac{b^3}{d^3}){r_1}^3 \tilde{r_1} + (\frac{3b^2}{2d^2} - \frac{3b^3}{2d^3}){r_1}^2 {\tilde{r_1}}^2 \\ & + (\frac{1}{6} - \frac{b}{d} + \frac{11b^2}{6d^2} - \frac{b^3}{d^3})\frac{{r_1}^4}{4} + \frac{b^3}{d^3}{\tilde{r_1}}^3 r_1 \end{aligned} \tag{3.37}$$

$$\begin{aligned} r_5 = {} & (\frac{b^2}{d^2} + \frac{3b^3}{d^3} + \frac{2b^4}{d^4}){r_1}^3 {\tilde{r_1}}^2 + (\frac{2b^3}{d^3} - \frac{2b^4}{d^4}){r_1}^2 {\tilde{r_1}}^3 \\ & + (\frac{b}{6d} - \frac{b^2}{d^2} - \frac{11b^3}{6d^3} - \frac{b^4}{d^4}){r_1}^4 \tilde{r_1} + \frac{b^4}{d^4} r_1 {\tilde{r_1}}^4 \\ & + (\frac{1}{24} - \frac{5b}{12d} + \frac{35b^2}{24d^2} - \frac{25b^3}{12d^3} + \frac{b^4}{d^4})\frac{{r_1}^5}{5} . \end{aligned} \tag{3.38}$$

Other expressions for r_6,r_7,..... can be obtained with this generalization as well. It is useful to arrange them in columns,

$$\begin{pmatrix} r_1 \to (\frac{b}{d}\tilde{r_1})^0 r_1\tilde{a} & 0 & 0 & \dots \\ r_2 \to (\frac{b}{d}\tilde{r_1})^1 r_1\tilde{a}^2 & (1-\frac{b}{d})\frac{{r_1}^2}{2}\tilde{a}^2 & 0 & \dots \\ r_3 \to (\frac{b}{d}\tilde{r_1})^2 r_1\tilde{a}^3 & 2(\frac{b}{d}\tilde{r_1})(1-\frac{b}{d})\frac{{r_1}^2}{2}\tilde{a}^3 & (1-\frac{b}{d})(\frac{1}{2}-\frac{b}{d})\frac{{r_1}^3}{3}\tilde{a}^3 & \dots \\ r_4 \to (\frac{b}{d}\tilde{r_1})^3 r_1\tilde{a}^4 & 3(\frac{b}{d}\tilde{r_1})^2(1-\frac{b}{d})\frac{{r_1}^2}{2}\tilde{a}^4 & 3(\frac{b}{d}\tilde{r_1})(1-\frac{b}{d})(\frac{1}{2}-\frac{b}{d})\frac{{r_1}^3}{3}\tilde{a}^4 & \dots \\ r_5 \to (\frac{b}{d}\tilde{r_1})^4 r_1\tilde{a}^5 & 4(\frac{b}{d}\tilde{r_1})^3(1-\frac{b}{d})\frac{{r_1}^2}{2}\tilde{a}^5 & 6(\frac{b}{d}\tilde{r_1})^2(1-\frac{b}{d})(\frac{1}{2}-\frac{b}{d})\frac{{r_1}^3}{3}\tilde{a}^5 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} . \quad (3.39)$$

The idea will be to resum each column separately. Denoting the sum of the m^{th} column by S_m, one finds from the first column

$$\begin{aligned} S_1 &= r_1\tilde{a} + (\frac{b}{d}\tilde{r_1})r_1\tilde{a}^2 + (\frac{b}{d}\tilde{r_1})^2 r_1\tilde{a}^3 + (\frac{b}{d}\tilde{r_1})^3 r_1\tilde{a}^4 + (\frac{b}{d}\tilde{r_1})^4 r_1\tilde{a}^5 + \dots \\ &= r_1\tilde{a}[1 + (\frac{b}{d}\tilde{r_1}\tilde{a}) + (\frac{b}{d}\tilde{r_1}\tilde{a})^2 + (\frac{b}{d}\tilde{r_1}\tilde{a})^3 + (\frac{b}{d}\tilde{r_1}\tilde{a})^4 + \dots] \\ &= r_1\tilde{a}(1 - \frac{b}{d}\tilde{r_1}\tilde{a})^{-1}. \end{aligned} \quad (3.40)$$

From the summation in the second column, we will have

$$\begin{aligned} S_2 &= (1-\frac{b}{d})\frac{{r_1}^2}{2}\tilde{a}^2[1 + 2(\frac{b}{d}\tilde{r_1}\tilde{a}) + 3(\frac{b}{d}\tilde{r_1}\tilde{a})^2 + 4(\frac{b}{d}\tilde{r_1}\tilde{a})^3 + \dots] \\ &= (1-\frac{b}{d})\frac{{r_1}^2}{2}\tilde{a}^2(1 - \frac{b}{d}\tilde{r_1}\tilde{a})^{-2}, \end{aligned} \quad (3.41)$$

and from the third column, we will get

$$\begin{aligned} S_3 &= (\frac{b^2}{d^2} - \frac{3b}{2d} + \frac{1}{2})\frac{{r_1}^3}{3}\tilde{a}^3[1 + 3(\frac{b}{d}\tilde{r_1}\tilde{a}) + 6(\frac{b}{d}\tilde{r_1}\tilde{a})^2 + 10(\frac{b}{d}\tilde{r_1}\tilde{a})^3 + \dots] \\ &= (\frac{b^2}{d^2} - \frac{3b}{2d} + \frac{1}{2})\frac{{r_1}^3}{3}\tilde{a}^3(1 - \frac{b}{d}\tilde{r_1}\,,\tilde{a})^{-3}, \end{aligned} \quad (3.42)$$

the fourth column will give

$$\begin{aligned} S_4 &= (\frac{1}{6} - \frac{b}{d} + \frac{11b^2}{6d^2} - \frac{b^3}{d^3})\frac{{r_1}^4}{4}\tilde{a}^4[1 + 4(\frac{b}{d}\tilde{r_1}\tilde{a}) + 10(\frac{b}{d}\tilde{r_1}\tilde{a})^2 + 20(\frac{b}{d}\tilde{r_1}\tilde{a})^3 + \dots] \\ &= (\frac{1}{6} - \frac{b}{d} + \frac{11b^2}{6d^2} - \frac{b^3}{d^3})\frac{{r_1}^4}{4}\tilde{a}^4(1 - \frac{b}{d}\tilde{r_1}\tilde{a})^{-4}, \end{aligned} \quad (3.43)$$

and from the fifth column

$$S_5 = (\frac{1}{24} - \frac{5b}{12d} + \frac{35b^2}{24d^2} - \frac{25b^3}{12d^3} + \frac{b^4}{d^4})\frac{{r_1}^5}{5}\tilde{a}^5$$

$$[1+5(\frac{b}{d}\tilde{r_1}\tilde{a})+15(\frac{b}{d}\tilde{r_1}\tilde{a})^2+35(\frac{b}{d}\tilde{r_1}\tilde{a})^3+\dots]$$
$$=(\frac{1}{24}-\frac{5b}{12d}+\frac{35b^2}{24d^2}-\frac{25b^3}{12d^3}+\frac{b^4}{d^4})\frac{{r_1}^5}{5}\tilde{a}^5(1-\frac{b}{d}\tilde{r_1}\tilde{a})^{-5}\,. \tag{3.44}$$

other columns will give us S_6, S_7, S_8,... . If we denote

$$S_1 = r_1\tilde{a}(1-\frac{b}{d}\tilde{r_1}\tilde{a})^{-1} = z\,, \tag{3.45}$$

then for the sum of the second column we obtain

$$S_2 = (1-\frac{b}{d})\frac{{r_1}^2}{2}\tilde{a}^2(1-\frac{b}{d}\tilde{r_1}\tilde{a})^{-2} = (1-\frac{b}{d})\frac{z^2}{2} = -(x-1)\frac{z^2}{2}\,, \tag{3.46}$$

where $x\equiv\frac{b}{d}$.

For S_3 we have

$$S_3 = (\frac{b^2}{d^2}-\frac{3b}{2d}+\frac{1}{2})\frac{{r_1}^3}{3}\tilde{a}^3(1-\frac{b}{d}\tilde{r_1}\tilde{a})^{-3} = (\frac{b^2}{d^2}-\frac{3b}{2d}+\frac{1}{2})\frac{z^3}{3}\,. \tag{3.47}$$

The term $(\frac{b^2}{d^2}-\frac{3b}{2d}+\frac{1}{2})$ can be factorized as

$$\frac{b^2}{d^2}-\frac{3b}{2d}+\frac{1}{2} = (\frac{-b}{d}+1)(\frac{-b}{d}+\frac{1}{2}) = (x-1)(x-\frac{1}{2})\,, \tag{3.48}$$

so

$$S_3 = (x-1)(x-\frac{1}{2})\frac{z^3}{3}\,. \tag{3.49}$$

For S_4

$$S_4 = (\frac{1}{6}-\frac{b}{d}+\frac{11b^2}{6d^2}-\frac{b^3}{d^3})\frac{{r_1}^4}{4}\tilde{a}^4(1-\frac{b}{d}\tilde{r_1}\tilde{a})^{-4} = (\frac{1}{6}-\frac{b}{d}+\frac{11b^2}{6d^2}-\frac{b^3}{d^3})\frac{z^4}{4}$$
$$=(\frac{-b}{d}+1)(\frac{-b}{d}+\frac{1}{2})(\frac{-b}{d}+\frac{1}{3})\frac{z^4}{4} = -(x-1)(x-\frac{1}{2})(x-\frac{1}{3})\frac{z^4}{4}\,, \tag{3.50}$$

and for S_5 we will have

$$S_5 = (\frac{1}{24}-\frac{5b}{12d}+\frac{35b^2}{24d^2}-\frac{25b^3}{12d^3}+\frac{b^4}{d^4})\frac{{r_1}^5}{5}\tilde{a}^5(1-\frac{b}{d}\tilde{r_1}\tilde{a})^{-5}$$
$$=(\frac{-b}{d}+1)(\frac{-b}{d}+\frac{1}{2})(\frac{-b}{d}+\frac{1}{3})(\frac{-b}{d}+\frac{1}{4})\frac{z^5}{5}$$
$$=(x-1)(x-\frac{1}{2})(x-\frac{1}{3})(x-\frac{1}{4})\frac{z^5}{5}\,. \tag{3.51}$$

Careful examination of the pattern of terms in Eq.(3.39) leads to the general result for S_m for $m>1$,

$$S_m = (-1)^{m+1}(x-1)(x-\frac{1}{2})(x-\frac{1}{3})+\dots+(x-\frac{1}{m-1})\frac{z^m}{m}\,. \tag{3.52}$$

Finally the resummed RG-predictable terms in the coefficient function will follow from $\mathcal{C} = 1 + S_1 + S_2 + S_3 + \ldots + S_n + \ldots$.

$$\begin{aligned}
\mathcal{C} = &= 1 + z - (x-1)\frac{z^2}{2} + (x-1)(x-\frac{1}{2})\frac{z^3}{3} - (x-1)(x-\frac{1}{2})(x-\frac{1}{3})\frac{z^4}{4} + \ldots \\
&= 1 + z - (x-1)\frac{z^2}{2!} + (x-1)(2x-1)\frac{z^3}{3!} - (x-1)(2x-1)(3x-1)\frac{z^4}{4!} + \ldots \\
&= 1 + z + x(\frac{1}{x}-1)\frac{z^2}{2!} + x^2(\frac{1}{x}-1)(\frac{1}{x}-2)\frac{z^3}{3!} + x^3(\frac{1}{x}-1)(\frac{1}{x}-2)(\frac{1}{x}-3)\frac{z^4}{4!} \\
&+ \ldots . \qquad (3.53)
\end{aligned}$$

Exchanging the $\frac{1}{x}$ by y gives us

$$\begin{aligned}
\mathcal{C} = &= 1 + z + y[y^{-2}(y-1)\frac{z^2}{2!} + y^{-3}(y-1)(y-2)\frac{z^3}{3!} \\
&+ y^{-4}(y-1)(y-2)(y-3)\frac{z^4}{4!} + \ldots] , \qquad (3.54)
\end{aligned}$$

and putting $y^{-1}z = w$ we obtain

$$\begin{aligned}
\mathcal{C} = &= 1 + z + [y(y-1)\frac{w^2}{2!} + y(y-1)(y-2)\frac{w^3}{3!} \\
&+ y(y-1)(y-2)(y-3)\frac{w^4}{4!} + \ldots] \\
&= z - yw + [1 + yw + y(y-1)\frac{w^2}{2!} + y(y-1)(y-2)\frac{w^3}{3!} \\
&+ y(y-1)(y-2)(y-3)\frac{w^4}{4!} + \ldots] . \\
&= (1+w)^y . \qquad (3.55)
\end{aligned}$$

Substituting for y,x,z and w yields

$$\mathcal{C} = \{1 + \frac{b}{d}[r_1\tilde{a}(1 - \frac{b}{d}\tilde{r_1}\tilde{a})^{-1}]\}^{\frac{d}{b}} = [\frac{1 - \frac{b}{d}\tilde{r_1}\tilde{a} + \frac{b}{d}r_1\tilde{a}}{1 - \frac{b}{d}\tilde{r_1}\tilde{a}}]^{\frac{d}{b}} . \qquad (3.56)$$

We can write the numerator in Eq.(3.56) as

$$(1 - \frac{b}{d}\tilde{r_1}\tilde{a} + \frac{b}{d}r_1\tilde{a}) = [1 + \tilde{a}b(\frac{r_1 - \tilde{r_1}}{d})] = (1 + \tilde{a}L) , \qquad (3.57)$$

where $L = b\ln(M/\mu) = b(r_1 - \tilde{r}_1)/d$. Since we are setting $c = c_2 = c_3 = \ldots = 0$ one has $(1 + \tilde{a}L)^{-1} = a/\tilde{a}$, substituting this into Eq.(3.56) gives

$$\mathcal{C} = [(1 - \frac{b}{d}\tilde{r_1}\tilde{a})\frac{a}{\tilde{a}}]^{\frac{-d}{b}} = [\frac{(1 - \frac{b}{d}\tilde{r_1}\tilde{a})}{\tilde{a}}a]^{\frac{-d}{b}} . \qquad (3.58)$$

Since $\tilde{a} = a(\mu) = 1/\tau$ we can rearrange Eq.(3.22) to obtain

$$\tilde{r_1} = \frac{d}{b}\frac{1}{\tilde{a}} - d\ln\frac{Q}{\Lambda_{\mathcal{M}}} , \tag{3.59}$$

and substituting this result into Eq.(3.58) we find

$$\mathcal{C} = \left(\frac{1}{b\ln(Q/\Lambda_{\mathcal{M}})}\right)^{d/b} a^{-d/b} . \tag{3.60}$$

Combining this with the anomalous dimension part $(ca)^{d/b}$ we reproduce the physical Q-dependence of $\mathcal{M}(Q)$ in Eq.(3.34).

3.5 Direct relation between Λ and QCD observables

As we shall discuss now it is possible to motivate the CORGI approach more straightforwardly by showing how QCD observables may be directly related to the dimensional transmutation parameter of the theory. We shall show how the dimensional transmutation parameter arises on the grounds of generalised dimensional analysis modifying the analysis of P.M.Stevenson [9]. The form of this relation at NLO level is then completely equivalent to the CORGI approach outlined above. The advantage of this derivation is that mention of the renormalization scale μ and the renormalised coupling $\alpha_s(\mu)$ can be essentially avoided and the physical irrelevance of these quantities is manifest. In contrast the fundamental importance of the parameter Λ is stressed. We suppose that we have a dimensionless generic QCD observable $R(Q)$ like (2.1), dependent on the single dimensionful (energy) scale Q. Quark masses will be taken to be zero throughout our discussion, the extension to the massive case has been considered in [30]. Since $R(Q)$ is dimensionless, we clearly must have , on the grounds of generalized dimensional analysis

$$R(Q) = \Phi(\frac{\Lambda}{Q}) , \tag{3.61}$$

where Λ is a dimensional scale, connected with the universal dimensional transmutation parameter of the theory, whose definition will depend on the way in which ultraviolet divergences are removed, $\Lambda_{\overline{MS}}$ for instance.

There is an extra trivial possibility that $R(Q) = C$, where C is a dimensionless constant. That is, there is no energy dependence. This trivial Q-dependence would be the case if the bare coupling of QCD was finite, since the QCD lagrangian (with massless quarks) contains no massive parameters. Of course, in fact, the bare coupling is infinite, and an infinite renormalization must be performed, leading to a functional relation as in Eq.(3.61).

The appearance of a dimensionful parameter such as Λ is due to the non-uniqueness of the theory. In QCD this is because the bare lagrangian corresponds to an infinite set of theories, each with a different Λ. Determination of the value of Λ, the one free parameter, can only be performed by experimental measurements. To fix the theory up to a one parameter degree of freedom we can specify the derivative of $R(Q)$ with respect to Q. The free parameter is now contained in the boundary condition imposed on integrating to obtain $R(Q)$. Effectively one is inverting Eq.(3.61) for $R(Q)$ to obtain

$$\frac{\Lambda}{Q} = \Phi^{-1}(R(Q)) \,. \tag{3.62}$$

where Φ^{-1} is the inverse function. This is indeed the basic motivation for Grunberg's Effective charge approach [13]. Following the argument of the subsection (2.4.4) we arrived at Eq.(2.65). To generalize the argument of that subsection we denote the integral $(\Delta\rho_0)$ by $G(R)$ so we have

$$b\ln(\frac{Q}{\Lambda_R}) = F(R) + G(R) \,. \tag{3.63}$$

The desired inverse function Φ^{-1} of Eq.(3.62) can then be obtained by exponentiating Eq.(2.65) which gives

$$\mathcal{F}(R(Q))\mathcal{G}(R(Q)) = \frac{\Lambda_R}{Q} \,, \tag{3.64}$$

where $\mathcal{F}$ is the universal function, defined by Eq.(2.68) and

$$\mathcal{G}(R) = e^{\frac{-G(R)}{b}} = exp\left[\int_0^{R(Q)} dx \left(\frac{-1}{\rho(x)} + \frac{1}{x^2(1+cx)}\right)\right] \,. \tag{3.65}$$

If only a NLO perturbative calculation has been completed then our state of knowledge of $\rho(x)$ is $\rho(x) = -bx^2(1+cx)$, since the NNLO and higher RS invariants ρ_2, ρ_3,... of

Eq.(2.51) will be unknown. From Eq.(3.65) we then have $\mathcal{G}(R)$=1 and the results of subsection 2.4.4 will be reproduced.

We need to relate the observable dependent Λ_R which arose on integrating Eq.(2.50) to the universal dimensional transmutation constant which depends only on the subtraction procedure used to remove the ultraviolet divergence, $\Lambda_{\overline{MS}}$ for instance. To see this we begin by noting that on rearranging Eq.(3.64) and taking the limit as $Q\to\infty$, we obtain an operational definition of Λ_R as in Eq.(2.67) or equivalently

$$\Lambda_R = \lim_{Q\to\infty} Q\exp\left(-\frac{F(R(Q))}{b}\right) , \tag{3.66}$$

we have used the fact that $G(0) = 0$ together with asymptotic freedom.
Recall that the coupling a satisfies the β-function equation

$$\mu\frac{\partial a}{\partial \mu} = \beta(a) = -ba^2(1 + ca + c_2a^2 + ...) .$$

If we define by $a(Q)$ the $\overline{MS}$ coupling with $\mu = Q$ we see that it will satisfy the β-function equation of the same form as in (2.50) for R with $\beta_{\overline{MS}}$ replacing $\beta(R)$. This may be integrated following the same step as above. The constant of integration Λ_R will be replaced by $\tilde{\Lambda}_{\overline{MS}}$ and the coefficient ρ_i by the $\overline{MS}$ β-function coefficients $c_i^{\overline{MS}}$. Again choosing $y = \infty$ with $K(x) = 0$ in (2.61) and (2.62) we arrive at.

$$\tilde{\Lambda}_{\overline{MS}} = \lim_{Q\to\infty} \exp\left(-\frac{F(a(Q))}{b}\right) . \tag{3.67}$$

From the perturbative expansion of R in (2.1) we will have

$$R(Q) = a(Q) + r(a(Q))^2 + ... \tag{3.68}$$

where we have defined for convenience $r\equiv r_1^{\overline{MS}}(\mu = Q)$, as the notation suggests r is Q-independent. It is then straightforward to show that as $Q \to \infty$

$$F(R) = F(a) - r + ... , \tag{3.69}$$

where the ellipsis denotes terms which vanish as $Q \to \infty$. Inserting this result into Eq.(3.67) one finally finds with using Eq.(3.66)

$$\Lambda_R = e^{\frac{r}{b}}\tilde{\Lambda}_{\overline{MS}} = (\frac{2c}{b})^{-\frac{c}{b}}\Lambda_{\overline{MS}}e^{\frac{r}{b}} , \tag{3.70}$$

for the promised exact relation between the observable-dependent and universal Λ's. The tilde over Λ is once again present to draw attention to the fact that the above choice of infinite integration constant in Eq.(2.61) with the standard choice [31]. This definition corresponds to translating the infinite constant by the finite shift $c\ln(\frac{b}{2c})$ (c.f. Eq.(2.41)).

In the CORGI approach the function $\mathcal{G}(R(Q))$ has the expansion

$$\mathcal{G}(\mathcal{R}(Q)) = 1 - \frac{X_2}{b}\mathcal{R}(Q) + O(\mathcal{R}^2) + \dots . \tag{3.71}$$

Here X_2 is the NNLO RS-invariant constant of integration which arises in Eqs.(2.76). Assembling all this we finally obtain the desired inverse relation between $\mathcal{R}$ and Λ, the universal dimensional transmutation parameter of the theory,

$$Q\mathcal{F}(\mathcal{R}(Q))\mathcal{G}(\mathcal{R}(Q))e^{-r/b}(2c/b)^{c/b} = \Lambda_{\overline{MS}} . \tag{3.72}$$

Notice that all dependence on the subtraction scheme chosen resides in the single factor $e^{-r/b}$, the remainder of the expression being independent of this choice. This corresponds to the observation of Celmaster and Gonsalves [18], that Λ's with different subtraction conventions can be exactly related given a one-loop (NLO) calculation. Recall that if only a NLO calculation has been performed $\mathcal{G} = 1$ since X_2 will be unknown, so that the best one can do in reconstructing $\Lambda_{\overline{MS}}$ is

$$Q\mathcal{F}(\mathcal{R}(Q))e^{-r/b}(2c/b)^{c/b} = \Lambda_{\overline{MS}} . \tag{3.73}$$

This is precisely the result obtained on inverting the NLO CORGI result $\mathcal{R} = a_0$ given by Eq.(2.83) as we demonstrate below.

First from (2.83) we have $a_0 = -\frac{1}{c[1+W(z(Q))]}$ where $z(Q) \equiv -\frac{1}{e}\left(\frac{Q}{\Lambda_{\mathcal{R}}}\right)^{-b/c}$ so with $\mathcal{R} = a_0$ we get

$$1 + W = \frac{-1}{c\mathcal{R}} \quad or \quad W = -(1 + \frac{1}{c\mathcal{R}}) , \tag{3.74}$$

we know from the definition of the Lambert-W function

$$W(z)e^{W(z)} = z \;\rightarrow\; -(1 + \frac{1}{c\mathcal{R}})e^{-(1+\frac{1}{c\mathcal{R}})} = z = -e^{-1}\left(\frac{\Lambda_R}{Q}\right)^{\frac{b}{c}} , \tag{3.75}$$

consequently

$$Q(1+\frac{1}{c\mathcal{R}})^{\frac{c}{b}}e^{-\frac{1}{b\mathcal{R}}}=\Lambda_R\,, \tag{3.76}$$

which using Eq.(3.70) yields

$$Q(1+\frac{1}{c\mathcal{R}})^{(\frac{c}{b})}e^{-\frac{-1}{b\mathcal{R}}}=e^{(\frac{r}{b})}\tilde{\Lambda}_{\overline{MS}}=e^{(\frac{r}{b})}(\frac{2c}{b})^{-\frac{c}{b}}\Lambda_{\overline{MS}}\,, \tag{3.77}$$

therefore

$$\Lambda_{\overline{MS}}=Q(1+\frac{1}{c\mathcal{R}})^{(\frac{c}{b})}e^{-\frac{-1}{b\mathcal{R}}}e^{(-\frac{r}{b})}(\frac{2c}{b})^{\frac{c}{b}}\,, \tag{3.78}$$

using Eq.(2.68) we obtain Eq.(3.73).

One can simply use Eq.(3.72) to test perturbative QCD. Given at least a NLO calculation for an observable $\mathcal{R}(Q)$ one simply substitutes the data values into Eq.(3.72), where $\mathcal{G}(\mathcal{R}(\mathcal{Q}))$ can include NNLO and higher corrections if known, and obtains $\Lambda_{\overline{MS}}$. To the extent that remaining higher-order perturbative and possible power corrections are small, one should find consistent values of $\Lambda_{\overline{MS}}$ for different observables. There is no need to mention μ or M in this analysis, let alone to vary them over an *ad hoc* range of values. For the moment problem the result corresponding to Eq.(3.72) is

$$Q\overline{\mathcal{F}}\left(\frac{\mathcal{M}}{A}\right)\overline{\mathcal{G}}\left(\frac{\mathcal{M}}{A}\right)e^{-\hat{r}/b}(2c/b)^{c/b}=\Lambda_{\overline{MS}}\,, \tag{3.79}$$

where $\hat{r}\equiv\tilde{r}_1^{\overline{MS}}(\mu=Q)$ is defined in Eq.(3.30). The modified functions $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ are most easily obtained by noting that $\hat{\mathcal{R}}$ in Eq.(3.28) is directly related to $\mathcal{M}/A$ and also satisfies Eq.(3.72). One finds

$$\begin{aligned}\overline{\mathcal{F}}(x)&=\exp\left[\frac{c}{b}(1-x^{-b/d})\right]x^{-b/d}\\ \overline{\mathcal{G}}(x)&=\left(1-\frac{X_2}{b}\frac{x^{b/d}}{c(1-x^{b/d})}+\ldots\right)\,.\end{aligned} \tag{3.80}$$

Where X_2 is the NNLO FRS-invariant which arises in Eqs.(3.24). The scheme-independent parameter A reflects a physical property of the operator $\mathcal{O}_n$ in Eq.(3.2). A_n and $\Lambda_{\overline{MS}}$ should be fitted simultaneously to the data for $\mathcal{M}_n(Q)$ using Eq.(3.79).

Here we referred only to some basic ideas for extracting $\Lambda_{\overline{MS}}$ in the CORGI approach for moments of SF and in order to complete the fitting procedure and get explicit results for $\Lambda_{\overline{MS}}$, further investigations are necessary that will be done as a new research activity in future work. In the next chapter we turn to a review of the large-order behaviour of perturbation theory, which will lead to discussion of renormalons. These techniques will be used to approximately resum the perturbative series to all-orders in the CORGI framework.

Chapter 4

Borel Transformation, Renormalon singularities and Adler D-function

4.1 Perturbation Theory in large-orders

The subject of large order perturbation theory has aroused new interest with particular attention paid to the power corrections to QCD predictions for hard scattering processes. We are theoretically interested to know how physical observables can be reconstructed from their power series expansion, whilst in most practical quantum field theories the perturbative expansions demonstrate divergent behaviour at large orders. This must cause concern since perturbation theory is our best tool for making predictions. Secondly, considerable effort has been devoted to the computation of higher-order QCD perturbative corrections, in some cases NNLO approximants are known, and we now seem to be at the limit of what can be achieved analytically or numerically. If the series is divergent, the next order may represent no improvement with respect to the lower order result.

The CORGI resummation of the last chapter, only worked because the series in which we were interested was convergent. Equally, we assumed that in the region where large infra-red logarithms were not a problem, truncating the perturbation series at fixed order was legitimate.

Within our discussion, we must clarify how it is that the application of fixed-order perturbation theory to QED and QCD has had such spectacular success when this technique appears to ignore an infinite number of unquantifable and divergent higher-order terms. We will attempt also to turn a knowledge of the large-order behaviour to our advantage and to use it to make contact with non-perturbative effects.

In this chapter we review the means by which perturbation theory, as applied to quantum field theory, has been investigated at large orders, and the insights and interpretation which has been consequently gained.

4.2 QED Vacuum instability and divergence of perturbation theory

The mechanism through which divergence of the perturbation series may occur in QED, was first presented by Dyson [32], where he argued that there is a singularity at the origin of the coupling constant complex plane, and that as a consequence, power expansions in the coupling are not analytical in the complex plane and are expected to diverge.

To find the reason for the divergence of QED perturbation theory, consider a generic perturbative expansion for a physical quantity f in QED as

$$f(e^2) = f_0 + f_1 e^2 + f_2 e^4 + ... = \sum_{k=0}^{\infty} f_k (e^2)^k \,, \tag{4.1}$$

where e is the electron charge and the coefficients f_k, after mass and charge renormalization are finite. Let us assume, for instance, that the series in Eq.(4.1) is an analytic function of e^2 as $e \to 0^+$, then we expect correspondingly that for small values of $-e^2$, $f(-e^2)$ will also be an analytic function with a convergent power series. Now we try to make a physical interpretation of $f(-e^2)$. It is well-known that in the physical world (where $e^2 > 0$), the interaction between two like charges is repulsive in nature. The

total energy of N electron-positron pair production procedure in the real world would be

$$E \sim NT + \frac{1}{2}e^2VN^2 \,, \tag{4.2}$$

where T is the mean kinetic energy and V the mean coulomb potential energy. The $\frac{N^2}{2}$ factor counts the number of interacting pairs. This system has a perfectly stable vacuum, as one increases N so the total energy of the system increases and the ground state of the system is achieved for N=0. On the other hand, in the fictitious world with $e^2 < 0$ we would have

$$E \sim NT - \frac{1}{2}e^2VN^2 \,. \tag{4.3}$$

As N increases E starts to increase until N reaches some critical value, N_{crit}

$$\frac{dE}{dN}|_{N=N_{crit}} = \left[T - e^2VN\right]|_{N=N_{crit}} = 0 \implies N_{crit} = \frac{T}{V \mid e^2 \mid}. \tag{4.4}$$

For $N > N_{crit}$, the total energy exhibits the behaviour

$$E \sim -N^2 \,, \tag{4.5}$$

which means that the "vacuum" state of such a fictitious world ($E \approx 0$) would not be the state of lowest energy and the vacuum would become catastrophically unstable.

Because of quantum barrier penetration effects, there is a finite probability for creation of N_{crit} pairs and, once this state is realised, an irreversible process of pair creation will set in until an infinite number of particle-antiparticle pair has been created, so the vacuum of this world would lead to an explosive creation of e^+e^- pairs and a never-ending decrease in the energy. Dyson [32] concluded that $f(e^2)$ could not be analytical for any $e^2 \leq 0$, and that, as a consequence, the QED perturbation series could not be convergent for a non-zero value of e^2. However as pointed out by P.M. Stevenson [9], the non-analyticity of a function does not necessarily imply non-convergence of its perturbation series expansion. It may simply signify that part of the function is not expandable in positive powers of the coupling, and that as such it is

non-perturbative in character.

In general we can say that a physical observable is determined by its perturbative expansion plus *non-perturbative effects*. For example, the term $e^{-\frac{1}{e^2}}$ is non-perturbative and would not contribute to the expansion on the right-hand side of Eq.(4.1)- it is invisible in perturbation theory. It is possible for the perturbative expansion to converge to some well-behaved analytic function and that it is the non-perturbative part of the function $f(e^2)$ that has an essential singularity at the origin. The latter is then wholly responsible for the singular nature of $f(e^2)$, and it is not possible to deduce anything about the divergence of the perturbative expansion from Dyson's argument.

Despite the fact that Dyson's analysis is not rigorous, the general conclusion that QED perturbative series have zero radius of convergence raises important considerations. We must explain exactly how truncated perturbative series still have meaning when the series themselves formally diverge for any physical value of the expansion parameter, and for this reason we need to introduce the concept of asymptotic series.

4.3 Asymptotic Series

In the example explained above, asymptotic series means that the sum on the right hand side of Eq.(4.1) does not necessairly reproduce $f(e^2)$ uniquely but rather it converges asymptotically towards $f(e^2)$ for a finite number of terms, after which it begins to diverge.

To express this more formally let us consider a general function $f(g)$ which is analytic in the domain D, where D is a subset of the complex plane having the origin as an accumulation point defined by:

$$D : |\arg\ g| \leq \tfrac{\theta}{2}, \quad |g| \leq \rho\,. \tag{4.6}$$

Perturbation theory is based on expressing $f(g)$ in the form of the power series at the

origin

$$f(g) = \sum_{k=0}^{\infty} f_k g^k \,, \tag{4.7}$$

where g is the expansion parameter (usually the coupling in QFT), is considered to be a small quantity. A formal series (4.7) is called asymptotic to $f(g)$ on the domain D if the sequence of *remainders* $R_N(g)$ satisfies [33],

$$R_N(g) = \mid f(g) - f_N(g) \mid \leq f_{N+1} \mid g \mid^{N+1} \quad \forall N, \forall g \in D \tag{4.8}$$

where $f_N(g) = \sum_{n=0}^{N} f_n g^n$ is the N^{th} partial sum. That is, the right-hand side of Eq.(4.7) diverges $\forall g \neq 0$ and in D it satisfies the bound Eq.(4.8) $\forall N$. So despite the fact that the infinite series is divergent for all non-zero values of g, the partial sum of N terms can be used to estimate $f(g)$ providing g is small.

The bound on $R_N(g)$ supplies the method by which we can make sense of a divergent series. It informs us that if we choose to truncate the series after the first N terms, we can obtain an approximation to $f(g)$ which for any N improves as $\mid g \mid \to 0$, with a discrepancy that is of one order higher then the approximant itself.

For a particular fixed value of $\mid g \mid$ we can minimize the bound on the error in this estimate by minimizing the right hand side of Eq.(4.8) with repect to N. Hence we can find the optimum number of terms, N_{opt}, to take in the partial sum. Up to N_{opt} terms the partial sum will converge, beyond it the series will begin to diverge. For QED the expansion parameter is the fine structure constant, $\alpha \simeq \frac{1}{137}$. Assuming the expansion coefficients grow as $n!$, minimizing $f_N a^N$ gives $N_{opt} \sim \frac{1}{\alpha} \sim 137$. Thus QED starts to diverge only at $\sim 137th$ order in the perturbative expansion, way beyond any order one could presently dream of calculating by summing Feynman diagrams.

A slightly more general form for the coefficients, which will lead to similar behaviour (that is, convergence up to N_{opt} terms and divergence beyond N_{opt} terms), is $f_N = Ac^{-N}N!$. We can then write down a function of N which will characterize the maximum

accuracy of the partial sum to N terms:

$$\epsilon(g) = min|_{\{N\}}(f_N \mid g \mid^N) = f_{N_{opt}} \mid g \mid^{N_{opt}} , \tag{4.9}$$

for f_N of the given form, $N_{opt} \sim \frac{c}{|g|}$. Using the Stirling formula for $N!$ at large N,

$$N! \sim \sqrt{2\pi N} e^{NlnN-N} , \tag{4.10}$$

we find that

$$\epsilon(g) \sim exp\left[\frac{-C}{\mid g \mid}\right] . \tag{4.11}$$

This finite accuracy means that the series is in fact asymptotic to an infinite number of functions and indicates that knowledge of the asymptotic series alone does not allow us to unambiguously and uniquely determine its corresponding function $f(g)$. If we have , for example

$$\sum_{n=0}^{\infty} f_n g^n \approx f(g) , \tag{4.12}$$

where '$\approx$' means '*is asymptotic to*', then it may be proved there always exists a function $f(g)$, analytic in the region $|\arg g| < \theta$ with $0 < \theta < \frac{\pi}{2}$, satisfying Eq.(4.12) for any set of coefficients $\{f_n\}$, however 'violent' the behaviour of the expansion coefficient f_n as $n \to \infty$.

We can add a term of the form $Aexp(\frac{-B}{g^n})$ with $0<2n\theta<\pi$ with A real and $B > 0$ without violating Eq.(4.8) within D, so

$$\sum_{n=0}^{\infty} f_n g^n \approx f(g) + Aexp(\frac{-B}{g^n}) , \tag{4.13}$$

provided $Bcos(\frac{\theta}{2}) > C$ and that $\mid A \mid$ is sufficiently small. This new function is also analytic in D and satisfies the bound given in Eq.(4.8). The ambiguity in Eq.(4.13) has the structure of a non-perturbative term. Thus the divergent nature of the perturbative series in itself implies that the perturbation theory cannot fully describe QFT. However, there is one situation in which the asymptotic series defines a unique function. If $\theta \geq \pi$ then, for some g such that $\mid \arg\ g \mid \geq \frac{\pi}{2}$, $Bcos(\arg\ g) \leq 0$ and the only

way in which the right hand side of Eq.(4.13) can be bounded by $\epsilon(g)$ is if $A = 0$. The right hand side of Eq.(4.13) reduces to $f(g)$ over the whole domain and the asymptotic series defines $f(g)$ uniquely.

The lesson we should learn from this is that we must not only investigate whether or not a perturbative series diverges but also, when it does diverge, if it still determines the relevant physical quantity uniquely. If we can define divergent series which do define physical quantities then we would like to find some way of reconstructing our generic $f(g)$ from its divergent expansion, and now we concentrate on how we might go about extracting useful information from divergent series. The question which must be addressed is how we can associate a function with a given asymptotic series. To do this we use the technique of Borel transformation.

4.4 Borel Summation

Consider the function (4.7) which has a perturbative series expansion

$$f(g) = \sum_{k=0}^{\infty} f_k g^k = \infty. \tag{4.14}$$

Although the original power expansion $f(g)$ makes no sense, it is possible to define a new series that has much improved convergence properties [34]. To see this let us divide each coefficient by $k!$ in order to obtain a more convergent series:

$$B[f](z) = \sum_{k=0}^{\infty} \frac{f_k}{k!} z^k \, . \tag{4.15}$$

We can then reconstruct $f(g)$ via

$$f(g) = \frac{1}{g} \int_0^{\infty} dz e^{\frac{-z}{g}} B[f](z) \, . \tag{4.16}$$

The function $B[f](z)$ is known as the Borel transform of $f(g)$ and, unlike $f(g)$ itself, it may have a non-zero radius of convergence (in the z-plane). If $B[f](z)$ exists and $f(g)$ can be reconstructed through Eq.(4.16) then we say that the series expansion of $f(g)$ is *Borel summable*. In general, for any physically interesting quantities, there will be

singularities in $B[f](z)$ and it will be our aim to isolate these and to find some method of regulating them such that we might be able to use Eq.(4.15) to extract some useful information about the function $f(g)$.

To motivate this method of summation we use a simple example. Consider the generic quantity $f(g)$ which has been calculated in perturbation theory with the coupling g. Using the definition of the factorial function this can be rewritten as

$$f(g) = \sum_{k=0}^{\infty} f_k g^k \frac{1}{k!} \int_0^{\infty} dt e^{\ t} t^n \ . \tag{4.17}$$

If the series (4.14) has a non-vanishing convergence radius δ, the integration (4.17) can be exchanged with the sum inside the circle. If we are outside the circle or if the convergence radius is zero, $\delta = 0$, we can exchange the order of integration and summation to define the series by the same expression, provided that the integral converges. In either case, taking $z = gt$ this gives

$$f(g) = \int_0^{\infty} dt e^{-t} \sum_{k=0}^{\infty} f_k \frac{(gt)^k}{k!} = \frac{1}{g} \int_0^{\infty} dz e^{\frac{-z}{g}} B[f](z) \tag{4.18}$$

where $B[f](z)$ is the Borel transfom of $f(g)$ defined in Eq.(4.15). So we could reproduce the function $f(g)$. The process of taking a divergent series in QFT and applying the Borel transform to extract a function which corresponds to it asymptotically is termed *Borel resummation*. This makes Borel summation a powerful tool in QFT.

The Borel method is also useful to control the singularities of the original series. This comes about because the convergence properties of (4.15) are better than those of (4.14) as can be seen by comparing the respective radii of convergence

$$\frac{1}{\rho_1} = \lim_{k\to\infty} \sqrt[k]{|\ f_k\ |}, \quad \frac{1}{\rho_2} = \lim_{k\to\infty} \frac{\sqrt[k]{|\ f_k\ |}}{k!}, \tag{4.19}$$

clearly if ρ_1 is non-vanishing, then ρ_2 will be infinite. Thus, singularities clustered at the origin will spread out in the complex plane on taking the Borel Transform.

The Borel summability of a perturbation series is not the same as the possibility of the complete determination of the physical observable from which the perturbative

expansion results. To deal with the problem of uniqueness in Borel summation we need to know more about the physical observable, and it turns out to be necessary to, a priori, prove something about the remainder function. The necessary criterion of Borel summability is given by Watson's lemma in [33] and later in [35]. If the full functional form of $f(g)$ may be recovered from its perturbative expansion we may call such a function *Borel recoverable* [36]. Since Watson's lemma implies that the function $f(g)$ is unique, we can conclude that no function of the form of Eq.(4.13) can satisfy the the condition (4.8) so the limiting of the remainder function is responsible for ruling out the possibility of non-perturbative effects. In general for QCD the conditions for Borel recoverability are not satisfied, which is to be expected since non-perturbative effects are known to be present.

In addition to uniqueness of Borel summable functions we need to know about their universality as well. The Green functions of a theory are related through the Schwinger-Dyson equations and are then constructed by the means of the operations- essentially made up of multiplications and divisions- acting on the Green functions. It is found that these operations do not result in new singularities in the Borel plane [37]. Therefore the interconnectedness through Schwinger-Dyson equations, of all Green's functions ensures that a singularity in the Borel transform of the one Green function propagates through to the Borel transform of all others. Moreover, it follows that the location of singularities in the Borel plane must be universal in all Borel transforms within each theory [38]. This implies the perturbative expansions of a theory all have similar divergence characteristics.

As a first example of an asymptotic series which can be Borel resummed, one may consider an alternating-sign factorial series such as $f_k = (-1)^k k!$ and consequently

$$f(g) = \sum_{k=0}^{\infty} (-1)^k k! g^k \,, \tag{4.20}$$

which has zero convergence radius (i.e it diverges for every $g \neq 0$), but we can assign a meaning to the sum of this series. The corresponding Borel transform and Borel sum

are then given by

$$B[f](z) = \sum_{k=0}^{\infty}(-z)^k = \frac{1}{1+z}, \quad f(g) = \int_0^{\infty} dz e^{\frac{-z}{g}} \frac{1}{1+z}, \tag{4.21}$$

where we have assumed analytic continuation along the whole real line. Fortunately, the singularity in the above integral sits outside the range of integration and therefore the integral exists. The method can fail if we take a fixed-sign factorial behaviour, $f_k = k!$ as an example, then we obtain

$$B[F](z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \quad F(g) = \int_0^{\infty} dz e^{\frac{-z}{g}} \frac{1}{1-z}. \tag{4.22}$$

The pole at $z = 1$ in the integration range means that this integral does not exist for a positive value of g (where $F(g)$ has a cut), nor is the Borel sum of such a series defined. For other values of g, the integral converges, selecting one of the functions having $\sum_{k=0}^{\infty} k! g^k$ behaviour as asymptotic expansion within the angle $0 <\arg\, g< 2\pi$. The summation can be defined in many ways, there are infinitely many functions with this asymptotic expansion. For instance, one can add any term of the form $Aexp(\frac{-1}{g})$ for $0 < g< \frac{1}{2}$.

With a slightly more general behaviour for the growth of the coefficients, we can consider,

$$f_k = {z_i}^k k! \,. \tag{4.23}$$

The Borel Transform is then of the form

$$B[F](z) = \sum_{k=0}^{\infty} \left(\frac{z}{z_i}\right)^k = \frac{1}{1-\frac{z}{z_i}}, \tag{4.24}$$

but we have a Borel sum which may not exist

$$F(g) = \int_0^{\infty} dz e^{\frac{-z}{g}} \frac{1}{1-\frac{z}{z_i}}. \tag{4.25}$$

because if $z_i > 0$ we will have a singularity at $z = z_i$ which will prevent the convergence of the integral for any positive value of the "coupling" g. In that case, one has to define a prescription to go around the pole. This can be done by shifting the pole from the real axis:

$$F(g) = \lim_{\epsilon \to 0} \int_0^{\infty} dz e^{\frac{-z}{g}} \frac{1}{1-\frac{z}{z_i} \pm i\epsilon} \pm i\pi e^{-\frac{z_i}{g}}. \tag{4.26}$$

In this integral the choice of the sign depends on whether one integrates above or below the positive semi-axis. The limit in (4.26) exists although the integral has no meaning, and it is called the *Cauchy principal value*. The residue (the imaginary part) gives us a measure of the ambiguity in the Borel sum. The ambiguity $\propto e^{-\frac{z_i}{g}}$, and as such it is heavily suppressed for small values of the"coupling" constant g.
On the other hand, if the large-order behaviour is assumed to be

$$f_k = {z_i}^{-n} n^\gamma n! \,, \tag{4.27}$$

with γ>0, the Borel transform is then of the form

$$B[f](z) \sim \frac{1}{(1-\frac{z}{z_i})^{\gamma+1}} \,. \tag{4.28}$$

For $\frac{z}{z_i} < 1$ we can expand binomially and if $\gamma \geq 0$ we obtain

$$B[f](z) \simeq \sum_{n=0}^{\infty} \frac{(n+\gamma)!}{n!\gamma!} \left(\frac{z}{z_i}\right)^n \,. \tag{4.29}$$

Using Stirling's theorem, one can then show that for large n

$$\frac{(n+\gamma)!}{n!\gamma!} = n^\gamma (1 + O(\frac{1}{n})) \,, \tag{4.30}$$

therefore to leading order in $\frac{1}{n}$ we obtain

$$B[f](z) \simeq \sum_{n=0}^{\infty} n^\gamma \left(\frac{z}{z_i}\right)^n \,. \tag{4.31}$$

We have seen that the Borel transform can be used to classify divergent series. The generic $n!$ divergence leads to a singularity in the Borel plane. The nature of this singularity is determined by the value of γ, if γ is positive we have a pole; and for non-integer γ a branch point singularity in the z-plane at $z = z_i$. In these examples the resulting ambiguity is of the order $exp(\frac{-z}{g})$ so that if this quantity is small, it may be neglected. For singularities more complicated than poles a similar exponentially small ambiguity arises near $g = 0$.

In dealing with the physical application of Borel transformation we need to get some insights into the large-N_f expansion, in which most field theories have a leading growth of the coefficient dominated by $n!$ [55], and this is why the Borel method will be of importance. We devote the next section to this subject.

4.5 The large-N_f expansion

The combinatorial growth in the number of Feynman diagrams at large orders was shown to be a strong enough reason for the coefficients of the ϕ^4 field theory to be factorially divergent [37, 39], and thus to give rise to singularities in the Borel transform. Since for simple theories like quantum mechanics and super-renormalizable field theories, the growth of the coefficients themselves is bounded by a power-law, this is expected to be the only source of divergences in these theories, and thus of singularities in the Borel transform. However, in renormalizable theories like ϕ^4 in four dimensions, QED, or QCD, there can be divergences arising from certain specific classes of diagrams. In fact diagrams containing chains of"bubbles". These *bubble chain diagrams* create a characteristic $n!$ divergence in the perturbative coefficients. Transforming to the Borel plane, this behaviour induces a set of singularities called *renormalons*. This feature was first identified in QED by Lautrup [39] and in QCD was discussed by t'Hooft [37]. As will be seen, it turns out that, in QCD , regions of high momentum are related to alternating-sign factorial growth of perturbative coefficients (ultraviolet of UV renormalons), and regions of small momentum are related to fixed-sign factorial growth (infrared or IR renormalons). We will begin our discussion of these singularities by looking at contributions to the QED vacuum polarization function and then turn to consider the analogous situation for the QCD vacuum polarization function.

The contribution from the vacuum polarization insertions to the renormalization of the gluon propagator consist of chains of "bubbles", such as the ones shown in the two types of diagrams in Fig.(4.1). The diagrams shown correspond to the leading contribution, since graphs of the type shown in Fig.(4.2) would be sub-leading in the coupling for a given number of "bubbles". Obviously, one has to sum over the contributions from chains with one "bubble", with two "bubbles", with three "bubbles", and so on, and thus one has a series. In QED, there would be only fermionic loops to consider, and these abelian "bubbles" would each contribute a numerical factor, proportional to N_f. In fact, a large-N_f approximation as an organizing principle was implied in the first studies of renormalons [39–41]. For further reference, we will define

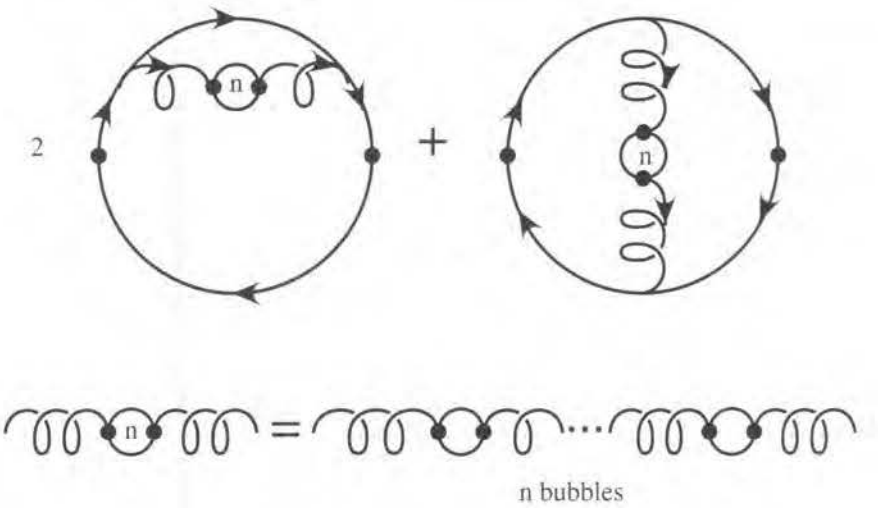

Figure 4.1: Leading large-N_f contribution at nth order in perturbation theory

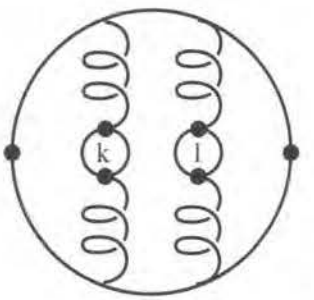

Figure 4.2: A generic two chain diagram

now the expansion of each perturbation coefficient in the leading-N_f expansion. For a wide range of "quark-initiated" [44] QCD observables, the perturbation coefficient d_n (c.f. Eq.(2.1)) can be written as

$$d_n = d_n^{[n]} N_f^n + d_n^{[n-1]} N_f^{n-1} + d_n^{[n-2]} N_f^{n-2} + ... + d_n^{[1]} N_f + d_n^{[0]}. \tag{4.32}$$

The N_f-expansion coefficients, will consist of sums of multinomials in $C_A = N$ and $C_F = (N^2 - 1)/2N$, of SU(N) QCD; and $d_n^{[n-r]}$ will have the structure $C_A^{n-r-s} C_F^s$ so that each term is of degree n.

The terms in the large N_f-expansion correspond to Feynman diagrams with differing numbers of vacuum polarization loops. In general, when the nth coefficient is

considered, the leading contribution $d_n^{[n]}N_f^n$ from the large -N_f expansion will come from the diagrams in Figure 4.1, since these diagrams contribute $\sim g^{2n+2}N_f^n \sim a^{n+1}N_f^n$. This contribution to the perturbative series at nth order from the leading coefficient $d_n^{[n]}N_f^n$ is found to behave like $n!$ for large n. This divergent behaviour of the expansion coefficents cannot be cancelled since these bubble diagrams have the highest power of N_f for any given power of the coupling a. We should note that these single bubble chains alone contribute the leading-N_f behaviour. For example, if we consider the behaviour of a generic two-chains diagram with k and l bubbles in the chain, see Figure 4.2, then this will contribute $g^{2k+2}g^{2l+2}N_f^{k+l} \sim a^{k+l+2}N_f^{k+l}$. We find this is $O(1/N_f)$ in the large-N_f expansion and hence sub-leading.

As we shall emphasise, it is actually more useful for our purpose to consider an expansion in powers of b. Although this expansion in b is convenient, we should emphasise that it does not have such a ready interpretation in terms of a class of Feynman diagrams as does the large-N_f expansion. To arrive at the standard leading-b expansion, one replaces $N_f \to \frac{11}{2}N - 3b$ obtaining

$$d_n = d_n^{(n)}b^n + d_n^{(n-1)}b^{n-1} + d_n^{(n-2)}b^{n-2} + ... + d_n^{(1)}b + d_n^{(0)} \ , \tag{4.33}$$

where $d_n^{(n)} = (-3)^n d_n^{[n]}$ exactly. One could equally well consider a "dual b-expansion" by replacing $N \to \frac{6}{11}b + \frac{2}{11}N_f$ in (4.32) to obtain an expansion in b with different coefficients. The standard "b-expansion" is exact in the large-N_f limit, and the "dual b-expansion" is exact in the large-N limit. Of course it is the standard expansion that is of practical use since we have all-orders large-N_f results.

The renormalons we shall be most concerned with, arise from one specific physical quantity which is related to the vacuum polarization function, called the *Adler D function*, and is the subject of the next section.

4.6 Adler D function

This function is related directly to measurable observables such as the R-ratio for e^+e^- annihilation into hadrons, which is defined by

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} . \tag{4.34}$$

We shall also consider R_τ, the total hadronic width for τ decay normalised to the leptonic width,

$$R_\tau \equiv \frac{\Gamma(\tau \to \nu_\tau + \text{hadrons})}{\Gamma(\tau \to \nu_\tau e^- \bar{\nu}_e)} . \tag{4.35}$$

The correlation function of two vector currents of massless quarks usually known as the *vacuum polarization function* is given by

$$(q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(s) = 4\pi^2 i \int d^4x e^{iq.x} < 0|T[j_\mu(x) j_\nu(x)]|0 > , \tag{4.36}$$

where $s = -q^2 > 0$ is the external euclidean momentum, and the vector-isovector j^μ is defined by

$$j^\mu \equiv \frac{1}{2}(: \bar{u}\gamma^\mu u : - : \bar{d}\gamma^\mu d :). \tag{4.37}$$

In order to avoid an unspecified constant, it is convenient to take a logarithmic derivative with respect to s and define the Adler D-function.

$$D(s) = -12\pi^2 s \frac{d}{ds}\Pi(s) . \tag{4.38}$$

R and Π may be related by using the optical theorem, which implies that

$$R(s) = 12\pi Im\Pi(-s)\}; . \tag{4.39}$$

In QCD perturbation theory we have

$$D(s) = N\sum_f {Q_f}^2 \left(1 + \frac{3}{4}C_F \tilde{D}(s)\right) + \left(\sum_f Q_f\right)^2 \tilde{D}(s) , \tag{4.40}$$

where Q_f denotes the electric charge of the quarks and the summation is over the flavours accessible at a given energy. The correction to the parton model result has the perturbative expansion

$$\tilde{D} = a + d_1 a^2 + d_2 a^3 + ... + d_n a^{n+1} + ..., \tag{4.41}$$

where $a = \frac{\alpha_s(\mu^2)}{\pi}$ is the renormalization group (RG) improved coupling. $\bar{D}$ denotes corrections of the "light-by-light" type which first enter at $O(a^3)$ and that are subleading in N_f. $\bar{D}$ is given by $\bar{D} = (\frac{55}{216} - \frac{5}{9}\zeta_3)a^3 + ...$, its contribution is numerically small. We shall ignore this term in our all-orders resummations. Our interest is in the asymptotic growth of the d_n coefficients in large-orders that can be expanded in terms of N_f as we explained in the last section.

We assume that each isolated QCD "bubble" will correspond to a factor of

$$-\frac{b}{2}\left[ln(\frac{k_E^2}{\mu^2}) + C\right] \equiv \Pi_0({k_E}^2)\ , \tag{4.42}$$

where $k_E^2 = -k^2$ is the euclidean virtual momentum carried by the chain and C is scheme-dependent. In the $\overline{MS}$ scheme, $C = -\frac{5}{3}$, we shall use the V-scheme, corresponding to the $\overline{MS}$ scheme with μ=exp($\frac{-5}{6}$)Q in the above expression so that $C = 0$. In next section we deal with the singularities which arise from renormalons relating to the Adler D-function.

4.7 Renormalon singularities

Following the argument of Parisi [45, 46], we can concentrate on the renormalization group equation (RGE) (c.f. Eq.(1.40)), satisfied by some generic n-point Green's function $\Gamma^{(n)}(p; a)$. He took a simple form for the β function, $\beta(a) = -ba^2$. Borel transforming the RGE gives

$$\left(-p\frac{\partial}{\partial p} - bz + n\right) B[\Gamma^{(n)}](p, z) = 0, \tag{4.43}$$

which has the solution

$$B[\Gamma^{(n)}](p, z) = r(z)p^n\left(\frac{p}{\mu}\right)^{(-bz)}\ . \tag{4.44}$$

As we have already taken care to point out, the Green's functions of a theory are interrelated by Schwinger-Dyson equations. As an illustration in considering the ϕ^4 theory, we can write a schematic form for the Schwinger-Dyson equation connecting

$\Gamma^{(6)}$ and $\Gamma^{(4)}$:

$$\Gamma^{(6)}(0,a) = \int \frac{d^4p}{(p^2+m^2)^3} \mid \Gamma^{(4)}(p,-p,0,0;a) \mid^3 , \tag{4.45}$$

we then write the Borel transform of this equation as

$$B[\Gamma^{(6)}(0,z)] \sim \int \frac{d^4p}{(p^2+m^2)^3} \left(\frac{p}{\mu}\right)^{(-bz)} , \tag{4.46}$$

where the factor $\left(\frac{p}{\mu}\right)^{(-bz)}$ comes from the expression in Eq.(4.44). The convergence of the integral in Eq.(4.46) depends crucially on the power of p in the integrand. For $B[\Gamma^{(6)}(0,z)]$ this p-dependence will be p^{-2-bz}, so for $z = \frac{-2}{b}$ the Borel transform will have a pole. The interdependences of Green's functions through Schwinger-Dyson equations such as Eq.(4.45) means that poles in the Borel transform of any Green's function will propagate to all others [37], giving the Borel transform poles at $z = z_l \equiv -\frac{2l}{b}$ where l is a positive integer. These poles which come directly from renormalization are known as we called them before, as renormalons. Divergences of this kind reflect the ultraviolet behaviour of the theory and are therefore called ultraviolet (UV) renormalons. In QED $b < 0$ and so the UV renormalons lie evenly spaced along the positive real z-axis. They represent a genuine ambiguity in the reconstruction of a function from its divergent perturbative expansion.

Parisi [45] showed that the location of UV renormalons is independent of the choice of β function, although their strengths have a weak dependence on the second β-function coefficient. Most importantly , he also derived a result which connected each pole with a local operator in the theory. This was based on the BPH theorem [2] which states that all UV divergences can be removed by the introduction into the lagrangian of counterterms of local operators.

The presence of UV renormalon poles in the Borel transform was first established for QED, a non-asymptotically free theory. The same results carry over to an asymptotically free theory but with one crucial difference. In SU(3) QCD with $N_f \leq 16$, for example, the first coefficient of the β-function is positive ($b = (11C_A - 2N_f)/6$) and

the UV renormalon poles now lie on the negative real z-axis. Thus these poles do not present any problem in Borel reconstruction of QCD quantities from their divergent perturbative expansions. However, one finds that a new set of poles appear on the positive axis. These are the infrared renormalons and they arise essentially from infrared divergences of bubble diagrams similar to that in Figure 4.1. In a procedure analogous to the BPH treatment of UV divergences in QED, Parisi [46] attempted to relate the IR renormalons to counterterms of non-local operators, a process about whose validity he himself expressed doubts. Nonetheless, significant progress has been made, notably by Grunberg [47] and Mueller [48], in relating IR renormalons to non-perturbative effects at low momenta through the operator product expansion (OPE). The question of how we deal with poles on the positive real axis is very important. At the very least we would like to know what degree of ambiguity these poles introduce into perturbative calculations, since large ambiguities would call into question the value of continuing the programme of extending fixed-order perturbative calculations of QCD observables. Less pessimistically, we might hope to obtain further insight into the non-perturbative regime of QCD.

There is indeed much to be learnt from studying the consequences of using a short-distance OPE to quantify some of the non-perturbative effects which we believe are intimately related to the presence of IR renormalons. The OPE, as proposed by Wilson [49] and implemented by Shifman, Vainshtein and Zakharov [50], enables one to attach some meaning to the limit in which the separation of two fields in coordinate space goes to zero. Formally, we write

$$T[\phi(x)\phi(0)] \sim \sum_i C_i(x)\mathcal{O}(O) \; . \tag{4.47}$$

Here the sum is over a set of local renormalized composite operators $\mathcal{O}$, and the C_i's, known as Wilson coefficients, are complex coefficient functions. Applying a short distance OPE to the Adler D-function $D(s)$ yields the representation

$$D(s) = \tilde{D}(a) + G_O(a) + \text{higher dimensional condensates} \; , \tag{4.48}$$

where the perturbative part of $D(s)$ ($\tilde{D}(a)$) has been defined by (4.41). $G_0(a)$ is the

leading (lowest dimensional) condensate contribution. For example, in QCD this will be the gluon condensate,

$$G_0(a) = \frac{1}{Q^4} < 0|GG|0 > (\mu) C_{GG}(Q/\mu, a) \,, \tag{4.49}$$

where $C_{GG}(Q/\mu, a)$ is the Wilson coefficient for the gluon condensate term in the OPE. In anticipation of the later calculations we shall assume massless quarks. Now we expect $\tilde{D}(a)$ and $G_0(a)$ to be separately RG invariant. If we consider a condensate with dimension $2l$ (that is, with scaling behaviour $\sim Q^{-2l}$) we can determine the a-dependence of $G_0(a)$ by requiring that $G_0(a)$ satisfies the renormalization group equation (see Eq.1.40). Let us take the following form for the QCD β-function

$$\beta(a) = \frac{da}{d\ln\mu} = -ba^2(1+ca) \,, \tag{4.50}$$

then we find the $G_0(a)$ must take the form

$$G_0(a) = C\left(\frac{\mu^2}{Q^2}\right)^l \exp\left[-\frac{z_l}{a}\right] a^\delta (1+O(a)) \sim Q^{-2l} \,, \tag{4.51}$$

where

$$\delta = \frac{2\gamma_0}{b} - cz_l \quad \text{where} \quad z_l = 2l/b \,, \tag{4.52}$$

with γ_0 the one loop anomalous dimension of the corresponding operator. C is a scale-independent constant which contains the truly non-perturbative information. Eq.(4.51) can be re-expressed in the form [47]

$$G_0(a) = \frac{Cz_l^{\delta-1}}{\Gamma(\delta)} \int_{z_l}^{\infty} dz \frac{e^{-z/a}}{(z/z_l - 1)^{1-\delta}} \,. \tag{4.53}$$

This form for $G_0(a)$ has an essential singularity at $a = 0$, so that the OPE motivated expression in Eq.(4.49) is only meaningful when we have a resummation prescription for $\tilde{D}(a)$. This will be provided by the Borel transform, in which the IR renormallon poles will be negotiated by performing the reconstruction integral along a contour displaced above or below the real z-axis.

Let us write $\tilde{D}(a)$ in its Borel representation

$$\tilde{D}(a) = \int_0^\infty dz e^{-z/a} B[\tilde{D}(a)](z) \,, \tag{4.54}$$

where the Borel transform is defined as before, as

$$B[\tilde{D}(a)](z) = \sum_{n=0}^{\infty} \frac{d_n}{n!} z^n \,. \tag{4.55}$$

Now let us assume that the first IR renormalon, arising from a bubble insertion as discussed above, occurs at $z = z_0$. By analogy with the case of UV renormalons, we can write down a form for $B[\tilde{D}(a)](z)$ in the region of this first renormalon pole.

$$B[\tilde{D}](z)_{z\simeq z_0}\, B[\tilde{D}_0](z) \equiv K \left(\frac{\mu^2}{Q^2}\right)^{bz_0/2} \frac{1}{(1-z/z_0)^\gamma} \left[1 + O\left(1 - \frac{z}{z_0}\right)\right] \,. \tag{4.56}$$

where K is a scale-independent factor. This yields (disregarding any UV renormalons on the negative real z-axis) the large order behaviour

$$d_k \sim K \left(\frac{\mu^2}{Q^2}\right)^{bz_0/2} \left(\frac{1}{z_0}\right)^\gamma \frac{\Gamma(n+\gamma)}{\Gamma(\gamma)} \left[1 + O\left(\frac{1}{n}\right)\right] \,. \tag{4.57}$$

As we did for $\tilde{D}(a)$ in Eq.(4.54), we can write the "renormalon contribution" to $\tilde{D}(a)$ in terms of its Borel transformation

$$\tilde{D}_0(a) = \int_0^\infty dz e^{-z/a} B[\tilde{D}_0(a)](z) \,. \tag{4.58}$$

For $z > z_0$ the $(1 - z/z_0)^\gamma$ factor in Eq.(4.56) implies that this contribution has an imaginary part (for $a > 0$). So

$$Im\tilde{D}_0(a) = \int_0^\infty dz e^{-z/a} B[Im\tilde{D}_0(a)](z) \,. \tag{4.59}$$

where

$$B[Im\tilde{D}_0](z) \equiv K \left(\frac{\mu^2}{Q^2}\right)^{bz_0/2} \frac{1}{(z/z_0 - 1)^\gamma} exp(\pm i\pi\gamma) \,. \tag{4.60}$$

Here we neglect terms of $O(z/z_0 - 1)$. The resulting ambiguity in the result of Eq.(4.60) depends on whether the contour is taken above or below the real z-axis. If this ambiguity is to be avoided we must obtain some compensating factor from the non-perturbative part of the theory. Due to the non-logarithmic UV divergences [51] C acquires an ambiguous imaginary part. We therefore assume the constant C in Eq.(4.53) is in fact complex. So we will have $C \to C_R \pm iC_I$, with the ambiguous imaginary part reflecting

the renormalon-induced ambiguity in $\tilde{D}(a)$. The critical step is now to require cancellation of these ambiguities between the parturbative Eq.(4.56), and non-perturbative, Eq.(4.53) parts of the theory. This leads to the following relations,

$$z_0 = z_l = 2l/b\,, \tag{4.61}$$

$$\delta = 1 - \gamma\,, \tag{4.62}$$

$$K = \frac{C_I {z_l}^{\delta-1}}{sin(\pi\delta\Gamma(\delta))}\,. \tag{4.63}$$

There are a number of important points to raise about these relations. First, the residue K is related to the condensate parameter, which leads us to expect that it is a process-independent quantity. The implication that one can obtain, by making an all-orders perturbative calculation, a "perturbation theory determined" part of the condensate contribution is misleading, in that exponentially small terms may be shifted between $\tilde{D}(a)$ and $G_0(a)$ [47]. Secondly Eq.(4.61) tell us the location of the IR renormalon poles on the positive real z-axis. The condensates obtained from performing a short distance OPE have dimensions 4,6,8,..., implying that the Borel transform for the Adler D-function will have IR renormalon singularities at $z_l = \frac{2l}{b}$ $(l = 2, 3, 4, ...)$. Notice that the lowest dimension condensate in the OPE is the gluon condensate with dimension 4. Hence we expect that there will be no IR renormalon at $z = 2/b$ since there is no dimension two condensate to compensate for it.

Turning now to a discussion about the renormalon singularities relating to the Adler D function, the contribution to the Adler function $\tilde{D}(a)$ from the sum over all the possible, multi-loop, one-chain diagrams such as the ones in Fig(4.1) is given by [52, 53]

$$\Pi(Q^2) = \sum_{k=0}^{\infty} a(Q^2) \int_0^{\infty} \frac{\delta\Pi(Q^2)}{\delta a(K_E^2)} \left[-\frac{b}{2} a(Q^2) ln(\frac{K_E^2}{\mu^2} e^C) \right]^k dK_E^2\,. \tag{4.64}$$

where Q is the external momentum and $a(Q^2)$ is either the one loop or two loop renormalised coupling. The kernel $\frac{\delta\Pi(Q^2)}{\delta a(K_E^2)}$ corresponds to the forward elastic scattering

amplitude of an off-shell vector-isovector quark current (of momentum Q) off an off-shell gluon (of momentum K_E) evaluated at the one loop level. With $\hat{k}^2 \equiv \frac{K_E^2}{Q^2}$, it is given by

$$\frac{\delta\Pi(Q^2)}{\delta a(K_E^2)} = \frac{N}{32\pi^2}\frac{C_F}{Q^2}\begin{cases} \hat{k}^2\Xi(\hat{k}^2), & \text{if } \hat{k}^2 \leq 1 \\ \frac{1}{\hat{k}^2}\Xi(\frac{1}{\hat{k}^2}), & \text{if } \hat{k}^2 \geq 1 \end{cases} , \tag{4.65}$$

where the IR$\rightleftharpoons$UV conformal symmetry ($\hat{k}^2 \rightleftharpoons \frac{1}{\hat{k}^2}$) is self-evident. This kernel was first mentioned in [52], and since one needs to include the appropriate colour factors to use it in QCD, it was recognised in [104] as being of utility in QCD. The function $\Xi(z)$ itself is defined as

$$\Xi(z) = 1 + \frac{4}{8}\int_0^z (1-\frac{y}{z})^2 \frac{\ln y}{1+y} dy \, , \qquad (z \geq 0) \, , \tag{4.66}$$

and the integral above can be explicitly calculated

$$\Xi(z) = \frac{4}{3}\frac{1}{z}\left\{1 - \ln z + (\frac{5}{2} - \frac{3}{2}\ln z)z + \frac{(1+z)^2}{z}[L_2(-z) + \ln z \ln(1+z)]\right\} \, , \tag{4.67}$$

where $L_2(x)$ is the dilogarithm function, defined as

$$L_2(x) = -\int_0^x \frac{\ln(1-y)}{y} dy \, . \tag{4.68}$$

To obtain the Adler function $\tilde{D}(a)$, one has to take the logarithmic derivative as defined in Eq.(4.38). As a result, the symmetry of the kernel is spoilt, and one has:

$$\begin{aligned} F^{IR}(\hat{k}^2) = \frac{N}{12\pi^2}\frac{C_F}{Q^2}\Big\{&(\frac{7}{4} - \ln\hat{k}^2)\hat{k}^2 + (1+\hat{k}^2) \\ &[L_2(-\hat{k}^2) + \ln\hat{k}^2 \ln(1+\hat{k}^2)]\Big\} \, , \qquad \hat{k}^2 \leq 1 \, , \end{aligned} \tag{4.69}$$

$$\begin{aligned} F^{UV}(\hat{k}^2) = \frac{N}{12\pi^2}\frac{C_F}{Q^2}\Big\{&1 + \ln\hat{k}^2 + (1+\hat{k}^2)[L_2(\frac{-1}{\hat{k}^2}) - \ln\hat{k}^2 \ln(1+\frac{1}{\hat{k}^2})] \\ &+ (\frac{3}{4} + \frac{1}{2}\ln\hat{k}^2)\frac{1}{\hat{k}^2}\Big\} \, , \qquad \hat{k}^2 \geq 1 \, . \end{aligned} \tag{4.70}$$

This function $F(\hat{k}^2)$ $\left(\equiv \frac{3}{8}\partial(\frac{\delta\pi}{\delta a})/\partial\hat{k}^2\right)$, considered together with the sum over chains, allows us to obtain an expression for the contribution of renormalons to the Adler function $\tilde{D}(a)$,

$$\tilde{D}(a(Q^2)) = \sum_{k=0}^{\infty} a(Q^2) \int_0^{\infty} F(\hat{k}^2) \left[-\frac{b}{2}aQ^2 \ln(\hat{k}^2 \frac{Q^2}{\mu^2} e^C)\right]^k d\hat{k}^2 \, . \tag{4.71}$$

This integral is most easily done by expanding $F^{IR,UV}(\hat{k}^2)$ in a power series

$$F^{IR}(\hat{k}^2) \propto \frac{3}{4}\hat{k}^2 - \frac{1}{2}(\frac{3}{2} - \ln\hat{k}^2)\hat{k}^4 + \frac{1}{6}(\frac{5}{6} - \ln\hat{k}^2)\hat{k}^6 + O(\hat{k}^6)\,, \tag{4.72}$$

$$F^{UV}(\hat{k}^2) \propto \frac{1}{6}\frac{\frac{5}{6} + \ln\hat{k}^2}{\hat{k}^4} - \frac{1}{12}\frac{\frac{7}{12} + \ln\hat{k}^2}{\hat{k}^6} + \frac{1}{20}\frac{\frac{9}{20} + \ln\hat{k}^2}{\hat{k}^8} + O(\frac{1}{\hat{k}^{10}})\,. \tag{4.73}$$

(the overall numerical factor was disregarded here). Obviously, these power expressions are only valid in the small-$\hat{k}$ and large-$\hat{k}$ limits. However if the renormalization scale is $\mu \equiv Q$, the leading contributions for the integral come indeed from the large logarithmic enhancements at $K_E \leq Q$ and $K_E \geq Q$, and the expressions above can be used in first approximation. We note with respect to the above expressions that the finiteness of the Adler function in both the infrared and ultraviolet is assured by the power-like structure of $F(\hat{k}^2)$. Also, $F^{IR}(\hat{k}^2)$ and $F^{UV}(\hat{k}^2)$ each have an infinite numbers of terms in their power series. We now break the integral (4.71) at $\hat{k}^2 = \frac{\mu^2}{Q^2}e^{-C}$ into two disjoint parts and perform the two integrations with the expansion (4.72) and (4.73) as the integrands. We obtain a series which consists of a linear combinations of alternating-sign and fixed-sign factorial terms [53]:

$$\begin{aligned}\tilde{D}(a(Q^2)) = \frac{N}{12\pi^2}\frac{C_F}{Q^2}\sum_{k=0}^{+\infty}\left(\frac{a(Q^2)}{2}\right)^{k+1}\Bigg\{&\frac{3}{4}(\frac{b}{2})^k(\frac{\mu^2}{Q^2}e^{-C})^2 - \frac{1}{9}(k+\frac{11}{2})(\frac{b}{3})^k(\frac{\mu^2}{Q^2}e^{-C})^3 \\ &+ \frac{1}{3}(k+\frac{11}{6})(-b)^k\frac{Q^2}{\mu^2}e^C - \frac{1}{24}(k+\frac{13}{6})(\frac{-b}{2})^k(\frac{Q^2}{\mu^2}e^C)^2\Bigg\}\,k!\,,\end{aligned} \tag{4.74}$$

where we considered only the two first terms in both Eq.(4.72) and Eq.(4.73). The Borel transform can be obtained straightforwardly (we choose the V-scheme at this point, $C = 0$):

$$\begin{aligned}B[D](Z) \propto &\frac{3}{4-bz} - \left[\frac{8}{(6-bz)^2} + \frac{\frac{11}{3}}{6-bz}\right] \\ &+\frac{1}{3}\left[\frac{8}{(2+bz)^2} + \frac{\frac{11}{3}}{2+bz}\right] - \frac{1}{3}\left[\frac{4}{(4+bz)^2} + \frac{\frac{13}{12}}{4+bz}\right]\,.\end{aligned} \tag{4.75}$$

In the first line of Eq.(4.75), we have Borel transform singularities on the positive real axis, which arise from the integration at low momenta, and are henceforth known IR renormalons; in the second line of Eq.(4.75), we have Borel transform singularities on the negative real axis, which arise from the integration of high momentum and henceforth known as UV renormalons. It is clear that consideration of more terms in the

power expansions of $F(\hat{k}^2)$ would lead to more singularities in the Borel transform, and hence more renormalons. The next ones would be at $z = \frac{8}{b}$ and $z = \frac{-6}{b}$. For the Adler functions, the series in (4.72) and (4.73) have no end, and an infinite number of IR_l and UV_l renormalons exists, located at $z_1 = \frac{-2}{b}$ and $z_l = \pm\frac{2l}{b}(l \geq 2)$.

The fact that there is no IR_1 renormalon can be traced back to the absence of a constant term in $F^{IR}(\bar{k}^2)$ and is related to the fact that there is no operator of dimension $\frac{1}{Q^2}$ in the OPE of the Adler function. Furthermore, since ,with the notable exception of IR_2, every (IR_l and UV_l) renormalon has a structure involving a $\ln \hat{k}^2$ and a number, multiplied by a (positive or negative) power of $\hat{k}^2$ (as can be seen in Eq.(4.72) and Eq.(4.73), respectively), one anticipates a pole + double pole structure for every generic singularity in the Borel sum of the Adler function. The ambiguties of IR renormalons, seem to be beyond perturbation theory. As we showed before tt is possible that these ambigutities will be compensated by the non perturbative power corrections associated with non-logarithmic UV divergences in coefficient functions.

An exact evaluation of the integral (4.71) is needed to obtain the full singularity structure in the Borel plane of the Adler function $\tilde{D}(a)$ in large-orders. To do this one may start by noting that $F(\hat{k}^2)$ can be written as a contour integral [53]

$$F(\hat{k}^2) = \frac{N}{6\pi^3}\frac{C_F}{Q^2}\sum_{k=2}^{\infty}(-1)^k\frac{d}{dk}\frac{1}{k^2-1} \int_{-\infty}^{+\infty} exp(irln(\hat{k}^2))\left(\frac{1+ir}{r^2+k^2} - \frac{1}{1-ir}\right)dr. \tag{4.76}$$

If we take the derivative in (4.76) and then do a trivial change of variables, we arrive at

$$F(\hat{k}^2) = \frac{iN}{32\pi^3}\frac{C_F}{Q^2}\int_{-\infty}^{+\infty} P(x)(\hat{k}^2)^{-x}dx, \tag{4.77}$$

where one has

$$P(x) = \frac{32}{3(1+x)}\sum_{k=2}^{\infty}\frac{(-1)^k k}{(k^2-x^2)^2}. \tag{4.78}$$

Now, if we invert (4.77) for $P(x)$, we obtain

$$P(x) = -\frac{16\pi^2}{N}\frac{Q^2}{C_F}\int_0^{\infty} F(\hat{k}^2)(\hat{k}^2)^{x-1}d\hat{k}^2. \tag{4.79}$$

We see that taking derivatives of $P(x)$ would bring factors of $ln(\hat{k}^2)$ to the integration above, effectively building the same integral as in (4.71). Thus $P(x)$ can be seen as a generating function for the large-orders coefficients of the Adler function $\tilde{D}(a)$. This result was obtained for the first time by D.J.Broadhurst [68].

The actual generating function for the coefficients of QCD Gell-Mann-Low function (MOM scheme β-function) is [68]

$$\Psi_n^{[n]} = \frac{3^{2-n}}{2}\frac{d^{n-2}}{dx^{n-2}}P(x)\mid_{x=1}, \tag{4.80}$$

which can be evaluated explicitly in closed form as [68]

$$\begin{aligned}\frac{\Psi_n^{[n]}}{(n-2)!} = \frac{n-1}{(-3)^{n-1}}[-2(2-n) - \frac{n+4}{2^n} \\ +\frac{16}{n-1}\sum_{\frac{n}{2}+1>s>0} s(1-2^{-2s})(1-2^{2s-n-2}\zeta_{2s+1})] .\end{aligned} \tag{4.81}$$

For the QCD Adler function in the $\overline{MS}$ scheme with $\mu = Q$ one has

$$d_n^{[n]} = 2T_f^n \sum_{m=0}^{n} \frac{(-5/9)^m \Psi_{n+2-m}^{[n+2-m]}}{m!(n-m)!}, \tag{4.82}$$

where the group theory factor is $T_f = \frac{1}{2}$ in the standard fermion representation. The $(\frac{-5}{9})^m$ factor enters since one is converting from the MOM scheme Adler function to that in $\overline{MS}$ scheme. Considerable simplification is achieved by using the V-scheme, the result (4.82) then becomes

$$d_n^{[n]} = 2{T_f}^n \Psi_{n+2}^{[n+2]} . \tag{4.83}$$

4.8 Leading-b expansion and RS-invariants

Even if the leading-b approximation is believed to work better at large orders, one can check its utility for the low orders perturbation coefficients for which numerical results are available. For $\tilde{D}$, the first two perturbative coefficients, d_1, d_2 are known from the exact perturbative calculation [59,64,100–102]. We shall assume $\overline{MS}$ renormalization

with renormalization scale $\mu = Q$ for the present, quoting the results, we have

$$\begin{aligned} d_1 &= (\frac{-11}{12} + \frac{2}{3}\zeta_3)N_f + C_A(\frac{41}{8} - \frac{11}{3}\zeta_3) - \frac{1}{8}C_F \ , \\ d_2 &= (\frac{151}{162} - \frac{19}{27}\zeta_3)N_f^3 + C_A(\frac{-970}{81} + \frac{224}{27}\zeta_3 + \frac{5}{9}\zeta_5)N_f + \\ & C_F(\frac{-29}{96} + \frac{19}{6}\zeta_3 - \frac{10}{3}\zeta_5)N_f + C_A^2(\frac{90445}{2592} - \frac{2737}{108}\zeta_3 - \frac{55}{18}\zeta_5) \\ & + C_A C_F(-\frac{127}{48} - \frac{143}{12}\zeta_3 + \frac{55}{3}\zeta_5) + C_F^2(\frac{-23}{32}) \ . \end{aligned} \tag{4.84}$$

For later comparisons it will be useful to write these results numerically for SU(N) QCD

$$\begin{aligned} d_1 &= -.115N_f + (.655N + \frac{.063}{N}) \ , \\ d_2 &= .086N_f^2 + N_f(-1.40N - \frac{.24}{N}) + (2.10N^2 - .661 - \frac{.180}{N^2}) \ . \end{aligned} \tag{4.85}$$

We now wish to demonstrate that the leading term in the b-expansion (Eq.(4.33)), when expanded in N_f, approximates the N_f-expansion coefficients well, even in rather low orders.

For d_1 and d_2 we have re-expanding the large-b term in powers of N_f,

$$\begin{aligned} d_1^{(1)}b &= .345b = -.115N_f + .634N \ , \\ d_2^{(2)}b^2 &= .776b^2 = .086N_f^2 - .948N_f N + 2.61N^2 \ . \end{aligned} \tag{4.86}$$

The subleading, N, $N_f N$, and N^2 coefficients approximate well in sign and magnitude the exact expressions in equations (4.85). The leading N_f and N_f^2 coefficents ,of course, agree exactly.

In fairness it should be noted that, whilst the leading-b term reproduces the subleading coefficents in the N_f-expansion at the ~20% level, there are significant cancellations between large terms and as a result the overall NNLO perturbative coefficients for $\tilde{D}$ are significantly overestimated by the leading-b term. For $N_f = 3$ and SU(3) QCD one has the exact ($\overline{MS}$, $\mu = Q$) coefficent $d_2 = 6.37$, to be compared with the leading-b term $d_2^{(2)}b^2 = 0.776b^2 = 15.7$.

Now to construct RS-invariant resummations the strategy will be to approximate the RS-invariants ρ_k in Eq.(2.57) and use Eq.(2.58), to obtain the approximate perturbative coefficients in any arbitrary RS. In this way invariance under the full RG transformations of QCD is guaranteed. For latter convenience we recall from Eq.(2.14) that $\rho_0 = \tau - d_1$ is RS-invariant. Therefore we can use d_1 itself, rather than τ, as was used in Eq.(2.58), to label the RS. Eq.(2.58) then become

$$\begin{aligned} d_2(d_1, c_2) &= d_1^2 + cd_1 + (\rho_2 - c_2) \\ d_3(d_1, c_2, c_3) &= d_1^3 + \frac{5}{2}cd_1^2 + (3\rho_2 - 2c_2)d_1 + \frac{1}{2}(\rho_3 - c_3) \\ &\vdots \quad \vdots . \end{aligned} \tag{4.87}$$

The result for $d_n(d_1, c_2, ..., c_n)$ is a polynomial of degree n in d_1 with coefficients involving ρ_n, ρ_{n-1},..., c and c_2, c_3,..., c_n; such that $d_n(0, \rho_2, \rho_3, ..., \rho_n)$=0. Given just these numbers the perturbative coefficients in any RS can be obtained from Eq.(4.87).

The ρ_k invariants can be organized as an expansion in b, with

$$\rho_k = \rho_k^{(k)} b^k + \rho_k^{(k-1)} b^{k-1} + \ldots + \rho_k^{(0)} + \rho_k^{(-1)} b^{-1} . \tag{4.88}$$

The $\rho_k^{(k)}$ can be obtained to all-orders from the large-N_f result for $d_k^{(k)}$ in Eq.(4.33). The b^{-1} term arises from the fact that in a 'regular' RS such as minimal subtraction the d_k are polynomials in b of degree k [99], whereas the corresponding β-function coefficients c_k are polynomials in b of degree $k-1$ with additional b^{-1} terms (c.f. the expression for $c = c_1$ in Eq.(1.59)). The RS-invariant combination in Eq.(4.88) in principle could contain arbitrary inverse powers of b, but RG considerations guarantee that only b^{-1} terms remain [99]. Thus $b\rho_k$ is a polynomial of degree $k+1$ in b.

As shown in references [42,63,69] the renormalon singularity structure leads to the expectation that the leading-b term when expanded in powers of N_f should, asymptotically, reproduce the sub-leading coefficients. That is, expressing $d_k^{(k)} b^k$ in Eq.(4.33) as

$$d_k^{(k)} b^k = \tilde{d}_k^{[k]} N_f^k + \tilde{d}_k^{[k-1]} N_f^{k-1} + \ldots + \tilde{d}_k^{[k-r]} N_f^{k-r} + \ldots + \tilde{d}_n^{[0]} , \tag{4.89}$$

one can show that

$$\vec{d}_k^{[k-r]} \sim d_k^{[k-r]}[1 + O(\frac{1}{k})] \,. \tag{4.90}$$

so that for fixed r and large k the sub-leading "N_f expansion" coefficients are reproduced. The leading-b expansion encodes the divergent behaviour of the perturbative coefficients at large orders. The effective charge β-function $\rho(D)$, Eq.(2.51), will contain Borel plane singularities at the same positions as those in Adler-D function [103] and hence one should expect a weak asymptotic result analogous to Eq.(4.90), with the $\rho_k^{(k)} b^{k+1}$ term asymptotically reproducing the N_f-expansion coefficients of $b\rho_k$. For the Adler D-function and DIS sum rules the level at which this works is again far in excess of that to be anticipated from the asymptotic result. The $\rho_k^{(k)}$ term involves only combinations of the $d_k^{(k)}$, with for instance $\rho_2^{(2)} = d_2^{(2)} - (d_1^{(1)})^2$, and so the $\rho_k^{(k)}$ can be obtained to all-orders given the exact leading-N_f all-orders results.

For the Adler D-function ($\tilde{D}$) one has the exact result for SU(N) QCD (where the "light-by-light" contribution $\bar{D}$" is excluded, see [58]

$$\begin{aligned} b\rho_2(\tilde{D}) = &-0.0243N_f^3 + (0.533N - 0.00151\frac{1}{N})N_f^2 + (-3.32N^2 + 0.344 + 0.0612\frac{1}{N^2})N_f \\ &+(3.79N^2 - 1.45N - 0.337\frac{1}{N}) \,. \end{aligned} \tag{4.91}$$

this to be compared with the "leading-b" piece

$$\begin{aligned} b^3\rho_2^{(2)}(\tilde{D}) = b^3(d_2^{(2)} - (d_1^{(1)})^2) = -0.656b^3 = -0.0243N_f^3 \\ +0.401NN_f^2 - 2.21N^2N_f + 4.04N^3 \,. \end{aligned} \tag{4.92}$$

Notice the good agreement of the sub-leading NN_f^2, N^2N_f, and N^3 coefficients. This gives us some confidence that the remarkable accuracy with which the sub-leading coefficients $d_k^{[k-r]}$ are reproduced is not just an artefact of the particular RS choice of $\overline{MS}$ with $\mu = Q$.

In the large-N limit (N_f=0) the RS-invariant ρ_2 is approximated to better than 10% accuracy. The 20% level agreement of the sub-leading coefficients does not, unfortunately, guarantee that the overall RS-invariant is reproduced to the same accuracy

for all N, N_f since there are large numerical cancellations. For instance for $N_f = 5$ one has $\rho_2(\tilde{D})$=-2.98 (exact), whereas $b^2\rho_2^{(2)}(\tilde{D})$=9.64.

In the next chapter we shall exploit the techniques relating to Borel transformation and Adler-D function to investigate some phenomenological results for the R-ratio and R_τ, and will consider scalar correlator function results for the Higgs decay width.

Chapter 5

Renormalon-inspired resummations, Estimating the uncertainty in $\alpha_s(m_\tau^2)$ and $\alpha(M_Z^2)$

5.1 Leading-b resummation for Adler-D function

The correlator of two vector currents in the Euclidean region is a fundamental ingredient in constructing a number of inclusive hadronic QCD observables of great importance in testing the theory. As we said before, by taking a logarithmic energy derivative of the correlator function, one can define the so-called Adler D-function, $D(s)$. By analytical continuation to the Minkowski region this quantity can then be directly related to the ratio , $R(s)$, of the total e^+e^- hadronic cross section to the point leptonic cross-section, and also to the analogous ratio R_τ of the total hadronic decay width of the τ lepton normalized to the leptonic decay width. The analytical continuation can be elegantly formulated as a contour integration of $D(s)$ together with a weight function around a circle in the complex energy s-plane [56, 57]. Performing this integration numerically with $D(s)$ approximated at some fixed order of perturbation theory then automatically resums to all-orders an infinite subset of potentially large analytical continuation terms involving powers of π^2 and beta-fuction coefficients, which arise in the running of the coupling around the integration contour [58]. These terms are usually truncated in the

direct fixed-order perturbative expansion of the Minkowskian quantity. Such approximations are referred to as "contour-improved".

The remaining uncertainty in these "contour-improved" predictions for R and R_τ comes from the uncalculated higher order terms in the perturbation series for $D(s)$, which has presently only been computed exactly to $O(\alpha_s^3)$ in the limit of massless quarks [59], and with some approximations including the contribution of top and bottom quarks [60]. We shall assume massless quarks in these investigations. As we discussed in the previous chapter, there are also effects due to non-perturbative power corrections [57, 61, 62]. We shall focus in this chapter on the former perturbative uncertainties. There are two interrelated aspects to these. As we discussed in Chapter 2, fixed order perturbation theory predictions have a dependence on the renormalization scheme (RS) chosen to define the coupling. In particular they depend on a dimensionful renormalization scale μ. Further, given a choice of RS there is an uncertainty due to the unknown $O(\alpha_s^4)$ and higher uncalculated perturbative terms. To estimate this one needs to perform a necessarily approximate all-orders resummation of these terms which is related to the large-N_f expansion that we discussed before. Recall that a well-motivated framework to accomplish this is provided by the so-called "leading-b" approximation [58, 63] (sometimes also referred to as "naive nonabelianization" [64-66]), which amounts to resumming to all-orders the portion of perturbative coefficients containing the highest power of b. This can be accomplished since in the large-N_f limit one has an exact all-orders result for the Adler D-function [54, 67, 68] as we discussed in Chapter 4. The leading-b resummation is then performed by replacing N_f by b. The strongest statement about the accuracy of this approximation that can be made, follows from an analysis of the operators which build the leading ultraviolet renormalon singularity [42, 69] (c.f. Section 4.8). One can prove that in the case of the vector Adler function the re-expansion of the leading-b result in powers of N_f correctly gives the asymptotics of the portion of perturbative coefficients proportional to ${N_f}^{n-r}N^r$ in n^{th}-order perturbation theory, with accuracy $O(1/n)$ [42] (c.f. Eq.(4.94)).

A remaining difficulty, first emphasised in [70], is the scale dependence of the leading-b resummations, if one tries to match them to the known *exact* NLO and NNLO perturbative coefficients. This matching ambiguity means that *any* result may be obtained by varying the scale . It was pointed out that claims [61,66] that comparison of fixed-order and "leading-b" resummed results indicated rather large uncalculated perturbative corrections for R_τ were undermined by this matching problem.

In this chapter, we wish to formulate the resummations in a very closely related, but technically much more straightforward way. Our plan is to perform a leading-b resummation for the Adler-D function in the CORGI approach (c.f. Chapter 2). As we shall see this is extremely straightforward to implement and the resulting resummed result can be written as a sum over exponential integral functions, representing the contributions of the ultraviolet and infra-red renormalons in the Borel plane. We begin the next section by reviewing the contour integral representation of $R(s)$ and R_τ in terms of $D(s)$, and describe a simple numerical algorithm for evaluating it.

5.2 Contour integral representation of Minkowski observables

We shall mainly be concerned in this chapter with the two inclusive QCD observables which we introduced in Section 4.6. The first one was the e^+e^- R-ratio, which has a perturbative part with expansion

$$\tilde{R}(s) = a(1 + \sum_{n>0} r_n a^n) \,, \tag{5.1}$$

and the second one, the ratio R_τ which was defined analogously as a ratio of the total τ hadronic decay width to its leptonic decay width as in Eq.(4.35), which has the form

$$R_\tau = N(|V_{ud}|^2 + |V_{us}|^2) S_{EW} \left[1 + \frac{5}{12} \frac{\alpha(m_\tau{}^2)}{\pi} + \tilde{R}_\tau + \delta_{PC} \right] \,. \tag{5.2}$$

with V_{ud} and V_{us} CKM mixing matrix elements with $|V_{ud}|^2 + |V_{us}|^2 \approx 1$. Since the energy scale $s = m_\tau^2$ lies below the threshold for charmed hadron production only

three flavours u,d,s, are active. The $\alpha(m_\tau^2)$ term denotes the leading QED electromagnetic corrections, and $S_{EW}\approx 1.0194$ [71] represents further electroweak corrections. δ_{PC} denotes possible power corrections. $\tilde{R}_\tau$ has a perturbative expansion

$$\tilde{R}_\tau = a(1 + \sum_{n>0} r_n^\tau a^n) \,. \tag{5.3}$$

As we saw before, both R and R_τ can be directly expressed in terms of the transverse part of the correlator of two vector currents in the euclidean region, and by taking a logarithmic derivative with respect to s which is external euclidean momentum, we defined the Adler D-function, that has the perturbative expansion

$$\tilde{D}(s) = a(1 + \sum_{n>0} d_n a^n) \,. \tag{5.4}$$

To relate the e^+e^- R-ratio to the QCD vaccum polarization function, we start by integrating Eq.(4.38)

$$\Pi(s) - \Pi(\hat{s}) = -\frac{1}{12\pi^2}\int_{\hat{s}}^{s} dt \frac{D(-t)}{t} \,, \tag{5.5}$$

where $\hat{s}$ is a reference time like momentum. Taking $s = Q^2$, $R(s)$ and $\Pi(s)$ may be related using the optical theorem supplemented by analyticity (c.f. Eq.(4.39))

$$R(s) = 12\pi Im\Pi(s + i\epsilon) = \frac{6\pi}{i}[\Pi(s + i\epsilon) - \Pi(s - i\epsilon)]. \tag{5.6}$$

Using Eq.(5.5) and Eq.(5.6) we can relate $\tilde{R}(s)$ and $D(s)$ succinctly by

$$R(s) = \frac{-1}{2\pi i}\int_{s-i\epsilon}^{s+i\epsilon} dt \frac{D(-t)}{t} \,. \tag{5.7}$$

Using Cauchy's theorm this can be converted into a contour integral running counter clockwise around the circle $t = s$ in the complex energy plane, cut along the positive real axis. Choosing $t = -se^{i\theta}$ the contour integral can be expressed as

$$R(s) = \frac{1}{2\pi}\int_{\pi}^{-\pi} d\theta D(se^{i\theta}) \,. \tag{5.8}$$

In order to evaluate R_τ as a contour integral, the theoretical analysis starts with the two-point correlation function for the vector $j^\mu_{ij,V} = \bar{\psi}_j\gamma^\mu\psi_i$ and axial-vector $j^\mu_{ij,A} =$

$\bar{\psi}_j\gamma^\mu\gamma_5\psi_i$, colour singlet quark currents (i,j=u,d,s), consequently one can define analogously to the vectorial current (4.37), an axial-vector current

$$A^\mu \equiv \frac{1}{2}(:\bar{u}\gamma^\mu\gamma_5 u: - :\bar{d}\gamma^\mu\gamma_5 d:) , \tag{5.9}$$

and a two-point correlation function for the axial-vector currents

$$\Pi_A^{\mu\nu}(q) \equiv i\int e^{iq.x} < 0|T\{A^\mu(x)A^\nu(0)^\dagger\}\mid 0 > d^4x , \tag{5.10}$$

then one has the Lorentz decompositions

$$\Pi_{V,A}^{\mu\nu}(q) = -(g^{\mu\nu}q^2 - q^\mu q^\nu)\Pi_{V,A}^{(1)}(q^2) + q^\mu q^\nu \Pi_{V,A}^{(0)}(q^2) , \tag{5.11}$$

where the superscript ($j = 0, 1$) denotes the angular momentum in the hadronic rest frame.

Since the imaginary parts of the two-point correlators are proportional to the spectral function for hadrons, the hadronic τ decay rate can be written as an integral of these functions over the invariant mass $s = -q^2 > 0$ of the final state hadrons [72]

$$R_\tau = 12\pi\int_0^{m_\tau^2}\frac{ds}{m_\tau^2}(1-\frac{s}{m_\tau^2})^2[(1+2\frac{s}{m_\tau^2})Im\Pi^{(1)}(s) + Im\Pi^0(s)], \tag{5.12}$$

with the correlator combinations according to the decomposition

$$\Pi^j(s) = |V_{ud}|^2(\Pi_{ud,V}^{(j)} + \Pi_{ud,A}^{(j)}) . \tag{5.13}$$

Here R_τ has been theoretically separated into contributions from specific quark currents, namely vector (V) and axial-vector (A) $\bar{u}d$-quark currents. Since we do not know how to account for the nonperturbative effects of QCD that bind quarks into hadrons, the integrand of Eq.(5.12) is not fully known. However, since the correlators are analytic function of s with a cut along the positive real s-axis, one can re-express (5.12) as a contour integral running counter-clockwise from $s = m_\tau^2 + i\epsilon$ to $s = m_\tau^2 - i\epsilon$

$$R_\tau = 6\pi i\int_{|s|=m_\tau^2}\frac{ds}{m_\tau^2}(1-\frac{s}{m_\tau^2})^2[(1+2\frac{s}{m_\tau^2})\Pi^{(1)}(s) + \Pi^{(0)}(s)] . \tag{5.14}$$

The advantage of this new expression is that knowledge of the correlators in the nonperturbative region $\mid s \mid \leq m_\tau^2$ is no longer required. Then, as the combination

$\Pi^{(0+1)}(q^2) = \Pi^{(0)}(q^2) + \Pi^{(1)}(q^2)$ must have a smoother limit than $\Pi^{(0)}(q^2)$, $\Pi^{(1)}(q^2)$ when $q \to 0$ (see Eq.(5.11)), we rewrite Eq.(5.12) as

$$R_\tau = 6\pi \int_0^{m_\tau^2} \frac{ds}{m_\tau^2}(1 - \frac{s}{m_\tau^2})^2[(1 + 2\frac{s}{m_\tau^2})Im\Pi^{(1+0)}(s) - 2\frac{s}{m_\tau^2}\Pi^0(s)] . \qquad (5.15)$$

In the limit of massless quarks, both vector and axial-vector currents are conserved so that $s\Pi^{(0)}(s) = 0$. Furthermore, the chiral invariance of massless QCD implies $\Pi^{0+1}_{ij,V}$=$\Pi^{0+1}_{ij,A}$=$\Pi(s)$, $i \neq j$. This implies that only the transverse correlator $\Pi^{(0+1)}(s)$ contributes to Eq.(5.15).

Replacing $\Pi(s)$ by the Adler D function and integrating (5.15) by parts yields

$$\tilde{R}_\tau = \frac{1}{2\pi i}\int_{|s|=m_\tau^2} \frac{ds}{s}(1 - 2\frac{s}{m_\tau^2} - 2\frac{s^3}{m_\tau^6} - \frac{s^4}{m_\tau^8})\tilde{D}(-s) , \qquad (5.16)$$

choosing the contour $s = -m_\tau^2 e^{i\theta}$ we obtain [73]

$$\tilde{R}_\tau = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta(1 + 2e^{i\theta} - 2e^{3i\theta} - e^{4i\theta})\tilde{D}(m_\tau^2 e^{i\theta}) . \qquad (5.17)$$

As a result of these considerations, we can say that a generic Minkowskian observable $\hat{R}(s_0)$ can then be related to $\tilde{D}(-s)$ by

$$\hat{R}(s_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Omega(\theta)\tilde{D}(s_0 e^{i\theta})d\theta , \qquad (5.18)$$

where $\Omega(\theta)$ is a weight function which depends on the observable $\hat{R}$. For $\Omega(\theta) = 1$ one has $\hat{R}(s_0) = \tilde{R}(s_0)$, and for $\Omega(\theta) = (1 + 2e^{i\theta} - 2e^{3i\theta} - e^{4i\theta})$ one has $\hat{R}(m_\tau^2)$=$\tilde{R}_\tau$. If one expands $\tilde{D}(s_0 e^{i\theta})$ as a perturbation series in $\bar{a}$≡$a(s_0 e^{i\theta})$ and numerically performs the θ integration term-by-term one obtains "contour-improved" perturbative results in which at each order an infinite subset of analytical continuation terms present in the conventional perturbation series of Eqs.(5.1), (5.3) are resummed. These terms are potentially large and involve powers of π^2 and beta-function coefficients, as is easily seen by expanding $\bar{a}$ in powers of $a(s_0)$ and integrating. In fact, to recover the conventional expression (5.1) for the perturbative $\tilde{R}(s)$, we insert the series (5.4) for $\tilde{D}(s)$ under the contour integral, and expand $a(se^{i\theta})$ in terms of $a(s)$,

$$a(se^{i\theta}) = a(s) - \frac{b}{2}i\theta a^2(s) + \left[-\frac{b}{2}i\theta + (\frac{-b}{2}i\theta)^2\right] a^3(s) + ... , \qquad (5.19)$$

evaluating the θ integrals, one obtains

$$r_1 = d_1, \;\; r_2 = d_2 - \frac{1}{12}\pi^2 b^2 \; , \tag{5.20}$$

the π^2 term arises due to the analytical continuation. Here we shall focus on this "contour-improved" version of perturbation theory. In Ref. [74] detailed comparisons of the performance of the two versions were made, and the importance of resumming the analytical continuation terms was emphasised.

An obvious numerical algorithm for evaluating the integral in Eq.(5.18) is to split the range from θ=0,π into K steps of size $\Delta\theta = \pi/K$ and perform a sum over the integrand evaluated at θ_n=$n\Delta\theta$, $n = 0, 1, ..., K$. So that

$$\hat{R}(s_0) \simeq \frac{\Delta\theta}{2\pi}[\Omega(0)\tilde{D}(s_0) + 2Re\sum_{n=1}^{K}\Omega(\theta_n)\tilde{D}(s_n)] \; , \tag{5.21}$$

where $s_n \equiv s_0 e^{in\Delta\theta}$. In practice we perform a Simpson's Rule evaluation. Writing the perturbation expansion for $\tilde{D}(s_n)$ we have

$$\tilde{D}(s_n) = \bar{a}_n + d_1\bar{a}_n^2 + d_2\bar{a}_n^3 + \ldots \; , \tag{5.22}$$

where we have defined $\bar{a}_n \equiv a(s_n)$. An efficient strategy [75] is to start with $\bar{a}_0 = a(s_0)$ and use Taylor's theorem step-by-step to evolve $\bar{a}_n$ to $\bar{a}_{n+1}$, using

$$\begin{aligned}\bar{a}_{n+1} = \bar{a}_n - i\frac{\Delta\theta}{2}bB(\bar{a}_n) - \frac{\Delta\theta^2}{8}b^2B(\bar{a}_n)B'(\bar{a}_n) + i\frac{\Delta\theta^3}{48}b^3[B(\bar{a}_n)B'(\bar{a}_n)^2 \\ + B(\bar{a}_n)^2B''(\bar{a}_n)] + O(\Delta\theta^4) + \ldots \; ,\end{aligned} \tag{5.23}$$

where $B(x)$ is the truncated beta-function The above use of Taylor's theorem is much faster to implement than the standard approach [61] of solving the integrated beta-function equation with complex renormalization scale s_n to find $\bar{a}_n$ at each step.

5.3 Fixed-order and resummed expressions for $D(s)$ in the CORGI approach

Recall that in the CORGI approach one avoids renormalization scale μ-dependence by performing a complete resummation of the ultraviolet logarithms which build the

dependence of the observable on the physical energy scale [76]. This is equivalent to directly relating the observable to the dimensional transmutation parameter of the theory [77], $\Lambda_{\overline{MS}}$ say. In this way one can write the CORGI series for $\tilde{D}(s)$,(c.f Eq.(3.25))

$$\tilde{D}(s) = a_0(s) + X_2 a_0^3(s) + X_3 a_0^4 + \ldots + X_n a_0^{n+1} + \ldots \,. \tag{5.24}$$

Here $a_0(s)$ is the CORGI coupling which may be written in terms of the Lambert W-function (defined implicitly by $W(z)\exp(W(z)) = z$ [22] as, $a_0(s) = -\frac{1}{c[1+W(z(s))]}$. $z(s) \equiv -\frac{1}{e}\left(\frac{\sqrt{s}}{\Lambda_D}\right)^{-b/c}$ where $\Lambda_D \equiv e^{d/b}(2c/b)^{-c/b}\Lambda_{\overline{MS}}$, with d the NLO perturbative coefficient d_1 for $\tilde{D}(s)$ in Eq.(5.4), in the $\overline{MS}$ scheme with $\mu^2 = s$. $a_0(s)$ is the coupling in the scheme with $\mu^2 = e^{-2d/b}s$, and the non-universal beta-function coefficients, c_i, $(i > 1)$ all zero. In this scheme $d_1 = 0$, and it is exactly equivalent at NLO to the Effective Charge approach of Grunberg [12, 13]. X_2 is the NNLO scheme-invariant combination (cf Eq.(2.78))

$$X_2 = c_2 + d_2 - cd_1 - d_1^2 \,, \tag{5.25}$$

built from the perturbative coefficients d_1 and d_2 and beta-function coefficients. The NLO and NNLO coefficients d_1 and d_2 are known exactly [59] (c.f. Eq.(4.84)) and so NNLO contour-improved CORGI predictions can be straightforwardly obtained for Minkowski observables $\tilde{R}(s_0)$, using the numerical integration described in Section 5.2. Since $a_0(s)$ is known in closed form in terms of the Lambert W-function, which has a well-defined branch structure in the complex plane, one can evaluate it directly, avoiding the Taylor's theorem trick in Eq.(5.23). In fact one needs to use the W_{-1} branch of the function (in the nomenclature of Ref. [78]) on the range of integration $[0, \pi]$, and the W_1 branch on the range $[-\pi, 0]$. The W function has many branches, only two of which are real; W_0 in the range $\frac{-1}{e} \leq x < \infty$ and W_1 in the range $\frac{-1}{e} \leq x < 0$. The requirement that the coupling is real and positive in the perturbative domain is sufficient to determine the branch to be W_1. As we shall discuss in Section 5.7 one can, in fact, avoid using the numerical Simpson's Rule integration all together for the case of the e^+e^- R-ratio where $\Omega(\theta) = 1$, and perform the integral in closed form in terms of logarithms of the W-function.

In order to assess the likely accuracy of the fixed-order perturbative approximation we can attempt to approximate the at present uncalculated coefficients d_i, $(i > 2)$ in $\tilde{D}(s)$ using the so-called "leading-b" approximation. In Eq.(4.33) the leading-b term $d_n^{(L)} \equiv d_n^{(n)} b^n$ is then used to approximate d_n. Since $d_n^{(L)} = (-3)^n d_n^{[n]} b^n$ it is known to all-orders. Using the exact large-N_f result one finds that the explicit all-orders result for $d_n^{(L)}$ in the V-scheme, following equations (4.81) and (4.83) is given by [68]

$$d_n^{(L)}(V) = -\frac{2}{3} n! \frac{(n+1)}{2^n} [-2n - \frac{n+6}{2^{n+2}} + \frac{16}{n+1} \sum_{\frac{n}{2}+1>s>0} s(1-2^{-2s})(1-2^{2s-n-2}\zeta_{2s+1})] b^n \,. \tag{5.26}$$

The resulting leading-b resummation

$$\tilde{D}^{(L)} = a(1 + \sum_{k=0}^{\infty} d_k^{(L)} a^k), \tag{5.27}$$

may then be defined as a principal value (PV) regulated Borel Sum,

$$\tilde{D}^{(L)}(1/a) = PV \int_0^{\infty} dz e^{-z/a} B[\tilde{D}^{(L)}](z) \,. \tag{5.28}$$

Here $B[\tilde{D}^{(L)}](z)$, denotes the Borel transform discussed in Chapter 4 which contains an infinite set of single and double poles at $z = z_l = \frac{2l}{b}$ corresponding to infra-red renormalons , IR_l, and an infinite set of ultra-violet renormalons, UV_l, at $z = -z_l$. The structure is

$$B[\tilde{D}^{(L)}](z) = \sum_{j=1}^{\infty} \frac{A_0(j) + A_1(j) z}{(1 + \frac{z}{z_j})^2} + \frac{B_0(2)}{(1 - \frac{z}{z_2})} + \sum_{j=3}^{\infty} \frac{B_0(j) + B_1(j) z}{(1 - \frac{z}{z_j})^2} \,. \tag{5.29}$$

The residues at these poles can be computed from the exact all-orders large-N_f result. The UV and IR renormalon contributions can then be easily expressed in terms of the exponential integral function,

$$Ei(x) = -\int_{-x}^{\infty} dt \frac{e^{-t}}{t} \,, \tag{5.30}$$

where for IR renormalons $x > 0$ and one defines $Ei(x)$ by taking the Cauchy principal value of the integral. As we showed in Eq.(4.26) the use of different regularizations gives an ambiguity of the same form as a power correction which should be small for small enough values of the coupling. Our particular choice of the Cauchy principal

value is simply for convenience. As we discussed in Chapter 4, the absence of a relevant operator of dimension two in the OPE for the vector correlator is in accord with the fact that the singularity IR_1 is not present, and the nearest singularity to the origin in the Borel plane is in fact UV_1, which generates the leading asymptotic behaviour [58],

$$d_n^{(L)}(V) \approx \frac{(12n+22)}{27} n! \left(-\frac{1}{2}\right)^n b^n \ . \tag{5.31}$$

One can then write the UV renormalon and IR renormalon contributions as infinite sums over the Ei functions,

$$\begin{aligned} \tilde{D}^{(L)}(F)|_{UV} = \sum_{j=1}^{\infty} z_j \{ & e^{F(a)z_j} Ei(-Fz_j)[Fz_j(A_0(j) - z_j A_1(j)) - z_j A_1(j)] \\ & + (A_0(j) - z_j A_1(j))\} \ , \end{aligned} \tag{5.32}$$

and

$$\begin{aligned} \tilde{D}^{(L)}(F)|_{IR} = & \ e^{-Fz_2} z_2 B_0(2) Ei(Fz_2) \\ & \sum_{j=3}^{\infty} z_j \{ e^{-Fz_j} Ei(Fz_j)[Fz_j(B_0(j) + z_j B_1(j)) - z_j B_1(j)] \\ & - (B_0(j) + z_j B_1(j))\} \ . \end{aligned} \tag{5.33}$$

Here we have defined $F \equiv 1/a_V$, where a_V is the coupling in the V-scheme. The $A_0(j), A_1(j)$ are related to the residues of the UV_j poles, with [58]

$$A_0(j) = \frac{8}{3} \frac{(-1)^{j+1}(3j^2+6j+2)}{j^2(j+1)^2(j+2)^2} \quad A_1(j) = \frac{4}{3} \frac{b(-1)^{j+1}(2j+3)}{j^2(j+1)^2(j+2)^2} \ . \tag{5.34}$$

Because of the conformal symmetry of the vector correlator (c.f. Section 4.7, Eq.(4.65)) the UV residues are directly related to the IR residues [79] with $B_0(j) = -A_0(-j)$ and $B_1(j) = -A_1(-j)$ for $j>2$, and $B_0(1) = B_1(1) = B_1(2) = 0$, and $B_0(2) = 1$ so IR_1 is absent, as required from the absence of a dimension two condensate in the OPE, IR_2 is a single pole and, all the other singularities are double poles. To evaluate the contour integral in the complex s-plane using this $\tilde{D}^{(L)}(F)$ result one needs to modify the definition of the Ei functions to cope with the fact that their argument involves $1/a_V(s_0 e^{i\theta})$ which is complex for nonzero θ. The appropriate generalization uses the function $Ei(n,z)$ defined by

$$Ei(n,z) = \int_1^{\infty} dt \frac{e^{-tz}}{t^n} \ . \tag{5.35}$$

This function is analytic everywhere in the cut complex z-plane, but has a branch cut along the negative real axis. One needs to replace $Ei(-Fz_j)$ in the UV contribution by $-Ei(1, Fz_j)$, and $Ei(Fz_j)$ in the IR contribution by $-Ei(1, -Fz_j) + i\pi sign(Im(Fz_j))$, where the discontinuity across the branch cut is removed by the final $i\pi$ contribution [58]. The final result for $\tilde{D}^{(L)}(F)$ is simply the sum of the UV and IR contributions. The sums in Eqs.(5.32),(5.33) are rapidly convergent since the $A(j)$ and $B(j)$ coefficients have a j^{-4} dependence for large j. For the numerical results to be reported in Section 5.4 we used $N_{UV} = 15$ and $N_{IR} = 17$ terms respectively in the two sums. It is sensible to arrange that $N_{IR} = N_{UV} + 2$, since the symmetry properties above mean that $A_0(j) = -B_0(j + 2)$, this ensures that the O(a) term in the perturbation series of Eq.(5.4) has the correct unit coefficient $B_0(2) = 1$.

The final step is to use the above results to perform an all-orders resummation in the CORGI approach. We would like to formally perform the resummation

$$\tilde{D}_{CORGI} = a_0 + X_2 a_0^3 + \sum_{n>2} X_n^{(L)} a_0^{n+1} , \tag{5.36}$$

so that the exactly known NNLO X_2 coefficient is included , with the remaining unknown coefficients approximated by $X_3^{(L)}, X_4^{(L)}, \ldots$, the leading-$b$ approximations. Note that a_0 is the full CORGI coupling , so that all the RG-predictable ultraviolet logarithms involving the exact NLO coefficient d_1 are completely resummed. This resummation is most easily achieved by noting that the combination [9] (c.f. Eq.(2.14))

$$\rho_0 = b ln\left(\frac{\mu}{\tilde{\Lambda}}\right) - d_1(\mu) , \tag{5.37}$$

is scheme-independent. At the leading-b level the coupling $a^{(L)}(s)$ is defined by the one-loop formula

$$a^{(L)}(s) = \frac{1}{b\ln(\sqrt{s}/\tilde{\Lambda})} . \tag{5.38}$$

In the CORGI scheme in leading-b approximation $d_1^{(L)} = 0$, and so by evaluating the invariant combination ρ_0 in Eq.(5.37) in the V-scheme and the CORGI scheme one can relate the couplings in the two schemes,

$$\frac{1}{a_V^{(L)}} = \frac{1}{a_0^{(L)}} + d_1^{(L)}(V) . \tag{5.39}$$

It then follows straightforwardly that the formal resummation in Eq.(5.36) is given by

$$\tilde{D}_{CORGI} = \tilde{D}^{(L)}\left(\frac{1}{a_0} + d_1^{(L)}(V)\right) + (X_2 - X_2^{(L)})a_0^3 \,, \tag{5.40}$$

in which the $\tilde{D}^{(L)}$ term is the all-orders sum with the exact X_2 replaced by $X_2^{(L)}$, and the second term corrects for this. One can obtain approximate N^{3}LO and higher CORGI results by truncating the sum in Eq.(5.36). The $X_n^{(L)}$ can be readily calculated by using the leading-b relation between the V-scheme and CORGI couplings in (5.39). One easily finds

$$X_n^{(L)} = \mathcal{C}_{n+1}\left[\sum_{k=0}^{\infty} d_n^{(L)}(V)\left(\frac{a}{1+ad_1^{(L)}(V)}\right)^{k+1}\right] \,, \tag{5.41}$$

where the symbol $\mathcal{C}_n[f(a)]$ denotes the coefficient of a^n in the power series expansion of $f(a)$. The $d_n^{(L)}(V)$ can be directly generated using the explicit result in Eq.(5.26).

Using the above results we can now straightforwardly generate all-orders resummed and fixed-order contour-improved CORGI results for the Minkowski observables R_τ and R. We shall perform some phenomenological studies in the next two sections.

5.4 Resummed versus fixed-order predictions for R_τ

The ratio R_τ defined by Eq.(4.35) has been the subject of a wide-ranging experimental study by the ALEPH collaboration [80]. If events involving strange quarks are removed from the data, they find $R_\tau = 3.492 \pm 0.016$. Setting $V_{us} = 0$, and $V_{ud} = 0.9754 \pm 0.0007$, and estimating the power correction contribution to be $\delta_{PC} = -0.003 \pm 0.004$ [80], one finds from Eq.(5.2) the experimental value $\tilde{R}_\tau = 0.2032^{+0.0160}_{-0.0159}$. The QED contribution has been neglected. One can then obtain all-orders leading-b resummed and fixed-order contour-improved CORGI results as described in Sections 2 and 3. We use $N_f = 3$ and fix $\Lambda^{(3)}_{\overline{MS}}$ so that the all-orders result reproduces the measured central value $\tilde{R}_\tau = 0.203$. The results are shown in Figure 5.1. The solid line is the all-orders resummed result fixed to the data, and the starred points show the N^{n}LO fixed-order CORGI results.

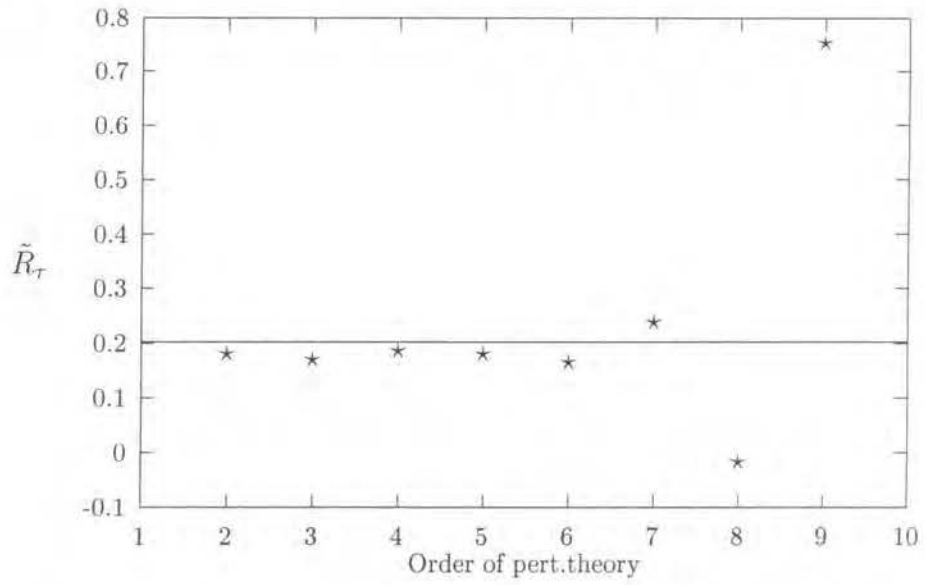

Figure 5.1: Fixed-order CORGI results for $\tilde{R}_\tau$ in N^{n}LO perturbation theory (starred points), compared to the all-orders resummation (solid line) fitted to ALEPH data.

We see that the NNLO (n=2) fixed-order result, which is the highest order exactly known, is in rather good agreement with the all-orders resummation. The leading-b approximated N^{n}LO results show an oscillatory trend which becomes explosive for $n > 7$, where fixed-order perturbation theory breaks down. The oscillatory behaviour is exactly what one would anticipate from the alternating-sign factorial growth of the contribution of the leading UV_1 renormalon, given by Eq.(5.31). To attempt to estimate the uncertainty in $\alpha_s(m_\tau^2)$ extracted from R_τ measurements we can use the difference between the resummed and exact NNLO fixed-order CORGI results to estimate the possible effects of uncalculated higher order terms. In Figure 5.2 we have plotted $\tilde{R}_\tau$ versus $\alpha_s(m_\tau^2)$. The upper solid curve is the all-orders CORGI result, whilst the lower dashed curve is the NNLO fixed-order CORGI result. We note that the separation of the curves increases rapidly with increasing $\tilde{R}_\tau$, so we are fortunate that for the experimentally measured $\tilde{R}_\tau \simeq 0.2$ the separation of the curves is reasonably small. Using the ALEPH data we find $\alpha_s(m_\tau^2) = 0.330^{+0.014}_{-0.013}$ from the all-orders CORGI result, and $\alpha_s(m_\tau^2) = 0.355^{+0.022}_{-0.022}$ from NNLO fixed order CORGI. The corresponding results which would have been obtained by integrating up the Effective Charge (EC)

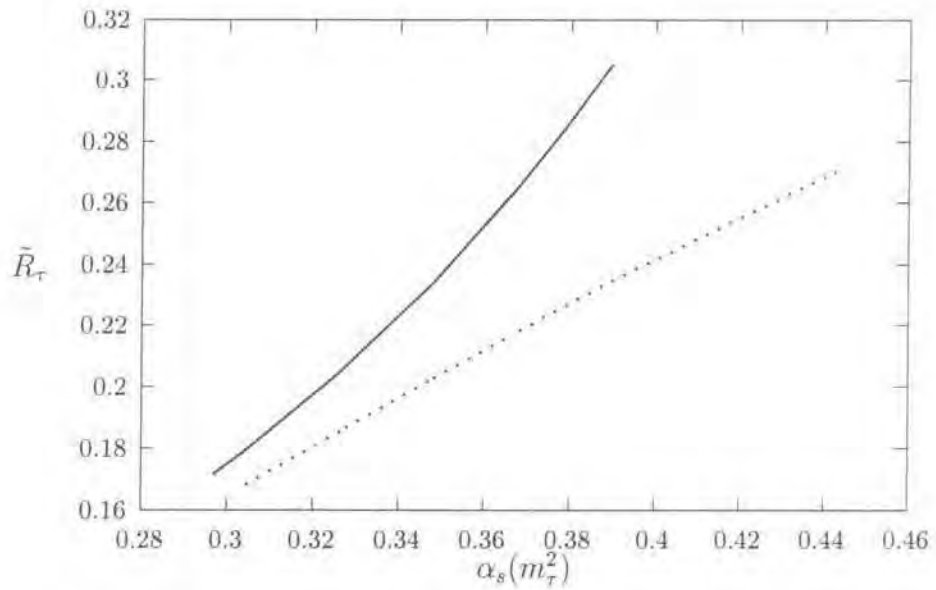

Figure 5.2: $\tilde{R}_\tau$ versus $\alpha_s(m_\tau^2)$. The dotted curve is the exact NNLO CORGI fixed-order result, and the upper solid curve is the approximate all-orders CORGI resummation.

beta-function for $\tilde{D}$ as in Ref. [74] are $\alpha_s(m_\tau^2) = 0.337^{+0.015}_{-0.016}$ and $\alpha_s(m_\tau^2) = 0.347^{+0.021}_{-0.022}$ for the resummed and NNLO EC results. So, as expected, the two approaches yield similar results.

We next need to evolve $\alpha_s(m_\tau^2)$ through flavour threshold to obtain $\alpha_s(M_Z^2)$. As we know the β-function governs the evolution of the strong coupling constant through the renormalization group (RG) equation (Eq.(1.40)). Contrary to what happens in the momentum-subtraction scheme (MOM) the $\overline{MS}$ β-function is quark mass independent. To obtain decoupling in the $\overline{MS}$ scheme we need to build in the decoupling region, $\mu \ll m$, with m the mass of the heavy particle, an effective field theory [81] that behaves as if only the light degrees of freedom were present. When the appropriate matching conditions are taken into account, the evolution of the strong coupling constant from low energies to high energies doesn't depend on the particular choice of the energy scale used to pass a heavy quark threshold. The residual dependence that appears in the perturbative calculation are just an estimate of the effects of the omitted higher-order corrections to the matching condition.

Here we used the recently calculated four-loop $\overline{MS}$ scheme QCD β-function to obtain the logarithmic pieces in the matching conditions. Matching conditions in QCD relate the strong coupling constant, a_{n_f}, in the full theory with n_f flavours, with the effective strong coupling constant a_{n_f-1} of the effective theory with $n_f - 1$ flavours through a power series in a_{n_f-1} [82]

$$a_{n_f}(\mu_{th}) = a_{n_{f-1}}(\mu_{th})[1 + \sum_{k=1}^{\infty} C_k(x) a^k_{n_{f-1}}(\mu_{th})] \,. \tag{5.42}$$

with the coefficents that depend on $x = log(\frac{\mu^2_{th}}{m^2})$ where m is some RG-invariant mass of the heavy quark (for instance the RG-invariant $\overline{MS}$ mass, $\bar{m}(\bar{m})$, or the perturbative pole mass M) that has been integrated out at the energy scale μ_{th}. In order to obtain a good approximation using only the first few terms in the perturbative expansion, we need to evaluate matching conditions in a region where $\mu_{th}/m \sim O(1)$, however the result of calculations should not depend on exactly which μ_{th} is chosen. Note that in order to simplify as much as possible the matching conditions a RG-invariant mass as reference mass, m, has been taken instead of the running mass $\bar{m}(\mu_{th})$ evaluated at the threshold scale μ_{th}. This makes matching conditions for the α_s's independent of the anomalous dimension.

Applying the RG equation (Eq.(3.4)) for β-function to both sides of Eq.(5.42) and identifying order-by-order the coefficents of $a_{n_{f-1}}$ we obtain for the C_k a set of coupled linear differential equations depending on only the β functions of the full and effective theories. We quote from ref [82] the results

$$\begin{aligned} C_1 &= \frac{x}{6}, \quad C_2 = c_{2,0} + \frac{19}{24}x + \frac{x^2}{36} \\ C_3 &= c_{3,0} + (\frac{241}{54} + \frac{13}{4}c_{2,0} - (\frac{325}{1728} + \frac{c_{2,0}}{6})n_f)x + \frac{511}{576}x^2 + \frac{x^3}{216}. \end{aligned} \tag{5.43}$$

If the RG-invariant $\overline{MS}$ mass is used as a reference scale, that is $m = \bar{m}(\bar{m})$, the coefficients one obtains are [83, 84]

$$c_{2,0} = -\frac{11}{72}, \quad c_{3,0} = \frac{82043}{27648} - \frac{575263}{124416} + \frac{2633}{31104}n_f. \tag{5.44}$$

Here, $c_{1,0} = 0$ which is the $\overline{MS}$ result with the usual dimensional regularization prescription, Tr{I}=4, with the trace taken in Dirac space.

In [85] is obtained an expression for the running in the strong coupling constant as an expansion in the strong coupling constant at one loop. Here that expression is improved by resumming some of the leading dependences proportional to c. This amounts to expanding around the approximate two-loop solution instead of the one-loop solution. At the required order we have

$$a(\mu) = a^{(2)}(\mu)(1 + c_2(\mu)[a^{(2)}(\mu)]^2 + c_3(\mu)[a^{(2)}(\mu)]^3) \,, \tag{5.45}$$

where $a^{(2)}$ is the approximate two loops solution

$$a^{(2)} = \frac{a(\mu_0)}{K + ca(\mu_0)L + c^2a^2(\mu_0)(1 - K + L)/K} \,, \tag{5.46}$$

and

$$\begin{aligned} c_2(\mu) &= c_2^{\overline{MS}}(1 - K) \\ c_3(\mu) &= \frac{c_3^{\overline{MS}}}{2}(1 - K^2) + cc_2^{\overline{MS}}K(K - 1 - L) + \frac{c^3}{2}(L^2 - (1 - K)^2), \end{aligned} \tag{5.47}$$

with $K = 1 + ba(\mu_0)ln(\frac{\mu}{\mu_0})$ and $L = lnK$. These expressions are convenient because the coupling constant at an arbitrary scale is given explicitly in terms of the coupling constant at some reference scale μ, which usually one takes equal to M_Z or m_τ.

Turning to the phenomenological application of the matching conditions and the four-loop running solution, we know that the most precise determinations of α_s are obtained from hadronic decays at LEP energies [86]. To compare these results with R_τ determination of $\alpha_s(m_\tau^2)$ one has to connect the strong coupling constant from a theory with three flavours at μ=m_τ to that in a theory with five flavours at a scale $\mu = M_z$. Therefore, two thresholds have to be passed, the threshold of the c-quark and the threshold of the b-quark. Note that, although $m_c < m_\tau$, results of $\alpha_s(m_\tau)$ are usually presented in a theory with only three quark flavours. This is because the mass of the highest charmed hadron is heavier than m_τ and so c-quarks only enter in loops.

Therefore it is appropriate to use an effective theory in which the c-quark has been integrated out.

In the following we obtained $\alpha_s(M_Z^2)$ by using as starting point $\alpha_s(m_\tau^2)$ and we estimated the residual errors due to the matching conditions and running for the different approximations used. At low energies, $\mu = m_\tau = 1777.0 \pm 0.3$ MeV, we know $\alpha_s^{(3)}(m_\tau^2)$, from this we can obtain $\alpha_s^{(3)}(\mu_{th}^{c\,2})$ at some matching point μ_{th}^c around $\bar{m}_c$ by using the renormalization group with $n_f = 3$,then ,by using Eq.(5.42) with $m = \bar{m}_c$ and $n_f = 4$ we obtain $\alpha_s^{(4)}(\mu_{th}^{c\,2})$. Now we use again the renormalization group with $n_f = 4$ to obtain $\alpha_s^{(4)}(\mu_{th}^{b\,2})$ at some matching point μ_{th}^b around $\bar{m}_b$ and use again Eq.(5.42) with $m = \bar{m}_b$ and $n_f = 5$ to obtain $\alpha_s^{(5)}(M_Z^2) = \alpha_s(M_Z^2)$. The final result will depend on the precise values used for μ_{th}^c and μ_{th}^b and this dependence gives an estimate of the errors which arise because of the truncation of the perturbative series in the matching conditions. In addition, matching conditions also depend on the masses of the quarks, and, although they are very well known, their actual value can effect the final result for $\alpha_s(M_Z^2)$. We use always as a reference scale the RG-invariant $\overline{MS}$ mass for quarks, and the coefficients in Eq.(5.43). For the quark masses we take the latest values in the literature, for the b quark mass $\bar{m}_b(\bar{m}_b) = 4.13 \pm 0.0066$ GeV [87]. For the c-quark mass we take $\bar{m}_c(\bar{m}_c) = 1.31 \pm 0.06$ GeV [88, 89].

In studying the effect of varying the scale at which matching is performed independently for the c and b quarks, on the central value of the strong coupling constant extracted from tau decay, it is found that varying the b-quark threshold scale in the range $\mu_{th}^b = 2 - 20$ GeV, four-loop running and three loop matching conditions conditions induce an error of 0.00009 on the strong coupling constant at the Z-boson mass scale. To study the errors induced in passing the c-quark threshold, $\mu_{th}^b = \bar{m}_b$ is fixed, and μ_{th}^c in the range $\mu_{th}^c = 1 - 4$ GeV is varied. Then an induced error of 0.0001 is found for four-loop running and three loop matching conditions. Finally using this evolution through flavour thresholds up to $\mu = M_Z$, we find $\alpha_s(M_Z^2)$=$0.120^{+0.002}_{-0.002}$ from the resummed CORGI result, and $\alpha_s(M_Z^2) = 0.123^{+0.002}_{-0.002}$ from the NNLO CORGI re-

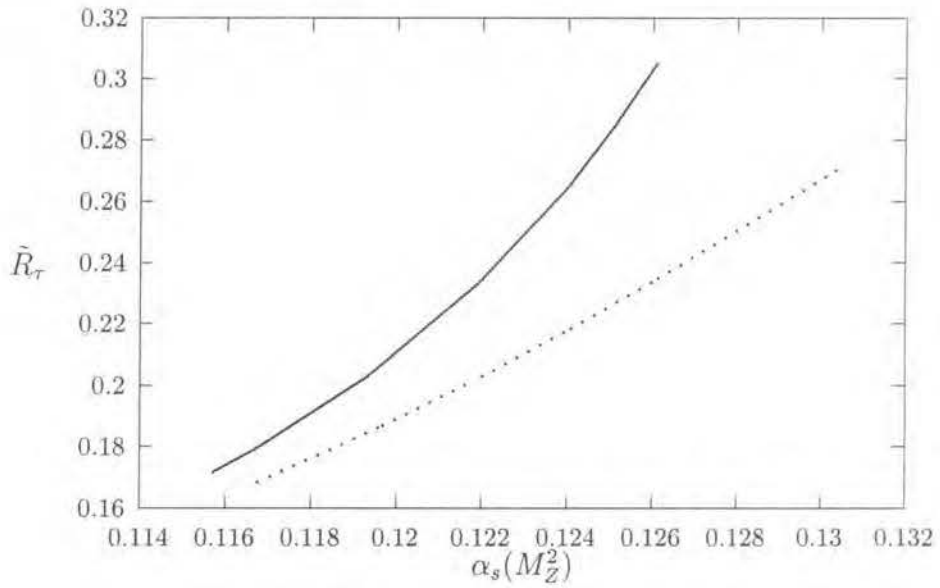

Figure 5.3: As Figure 2 but versus $\alpha_s(M_Z^2)$.

sult. Thus, we conservatively estimate an uncertainty $\delta\alpha_s(M_Z^2)\approx 0.003$. A direct plot of the resummed and NNLO results for $\tilde{R}_\tau$ versus $\alpha_s(M_Z^2)$ is given in Figure 5.3.

The invariant mass distribution of the produced hadrons in τ decay is well-measured experimentally [80, 90]. We define the quantity $R_\tau(s_0)$ as

$$R_\tau(s_0) \equiv \frac{\Gamma(\tau \to \nu_\tau + \text{hadrons}; s_{had} > s_0)}{\Gamma(\tau \to \nu_\tau e \bar{\nu}_e)} = \int_0^{s_0} ds \frac{dR_\tau(s)}{ds} , \tag{5.48}$$

where $\frac{dR_\tau}{ds}$ denotes the measured inclusive hadronic spectrum.

$$R_\tau(s_0) = N(|V_{ud}|^2) S_{EW} [(2x - 2x^3 + x^4) + \frac{3}{4} c_F \tilde{R}_\tau(s_0) + \delta_{PC}] , \tag{5.49}$$

with $x \equiv s_0/m_\tau^2$. The perturbative part $\tilde{R}_\tau(s_0)$ can be computed from Eq.(5.18) with the choice of weight function

$$\Omega(\theta) = 2x(1 + e^{i\theta}) - 2x^3(1 + e^{3i\theta}) + x^4(1 - e^{4i\theta}) . \tag{5.50}$$

It is then straightforward to obtain contour-improved fixed-order and resummed CORGI results for $\tilde{R}_\tau(s_0)$. In Figure 5.4 we show the fit of the all-orders leading-b CORGI resummation (solid line) to the ALEPH data for $R_\tau(s)$ (open circles) [80]. The resummation is fitted to the data at $s = m_\tau^2$, where $R_\tau(m_\tau^2) = R_\tau$. The CORGI

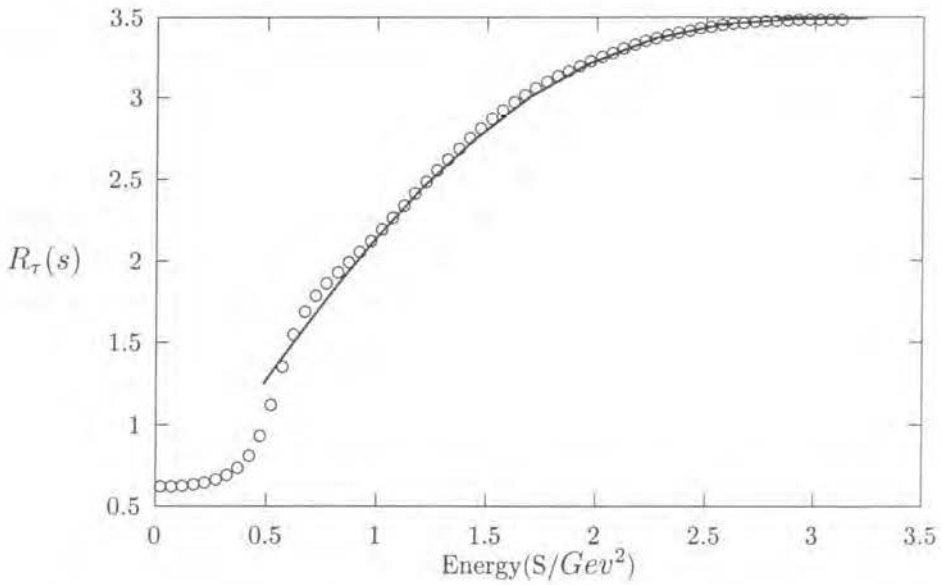

Figure 5.4: ALEPH data for $R_\tau(s)$ (open circles) compared with leading-b all-orders CORGI result fitted at $s = m_\tau^2$ (solid curve).

coupling has a Landau pole at $\sqrt{s} = \Lambda_D$, as is apparent from Eq.(2.83). Fitting to the experimental value of R_τ determines $\Lambda_D = 0.572$ GeV, and so the resummed prediction is only defined for $s > 0.327$ GeV^2. There is excellent agreement with the data. On this scale the fixed-order NNLO CORGI result would not be distinguishable from the all-orders result, and so we have not included it on the plot.

5.5 Estimating the uncertainty in hadronic corrections to $\alpha(M_Z^2)$

In this section we wish to make use of the difference between the NNLO fixed-order and resummed CORGI results for $\tilde{R}(s)$ in e^+e^- annihilation to estimate the uncertainty in $\alpha(M_Z^2)$, the QED coupling at the Z pole, which plays a crucial role in constraining the Standard Model Higgs mass from precision electroweak fits to radiative corrections [91]. We begin, however, by plotting some figures, analogous to Figure 5.1, to indicate the performance of fixed-order perturbation theory versus the resummed results at various energies. In Figure 5.5 we show the all-orders CORGI leading-b resummation (solid

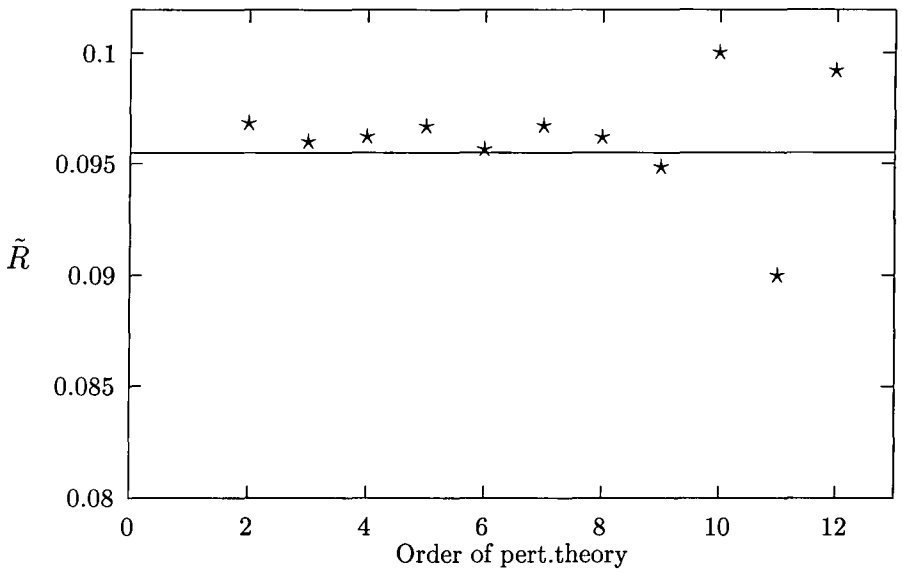

Figure 5.5: Fixed order results (starred points) for $\tilde{R}$ versus different orders of perturbation theory at $\sqrt{s} = m_\tau$=1.777 GeV. The solid line shows $\tilde{R}$ for the all-orders contour-improved resummation.

line) and fixed order results (starred points) for $\tilde{R}(s)$ at $\sqrt{s} = 1.777$ GeV, corresponding to m_τ, so the performance can be directly compared to Figure 5.1. The only difference in the two calculations is the choice of weight function, $\Omega(\theta)$ in Eq.(5.18). The oscillatory trend due to the leading ultraviolet renormalon is again evident, with wild oscillations setting in at $n > 9$ where fixed-order perturbation theory breaks down.

In Figure 5.6 we present a corresponding plot at LEP1 energy $\sqrt{s} = M_Z$. Clearly at the higher energy the agreement is much improved. With the fixed-order results exactly tracking the all-orders result for $n > 4$. Wild oscillations only set in for $n > 30$ at this higher energy. Finally, in Figure 5.7 we show a plot of $\tilde{R}(s)$ versus $\ln(\sqrt{s}/\text{GeV})$, in the range $1 < \sqrt{s} < 91$ GeV. The solid line corresponds to the all-orders resummed result and the dashed line to the NNLO fixed-order CORGI result. We assume $\alpha_s(M_Z^2) = 0.119$, and evolve through flavour thresholds using the three-loop matching condition [82, 83]. The QED fine structure constant is extremely well-

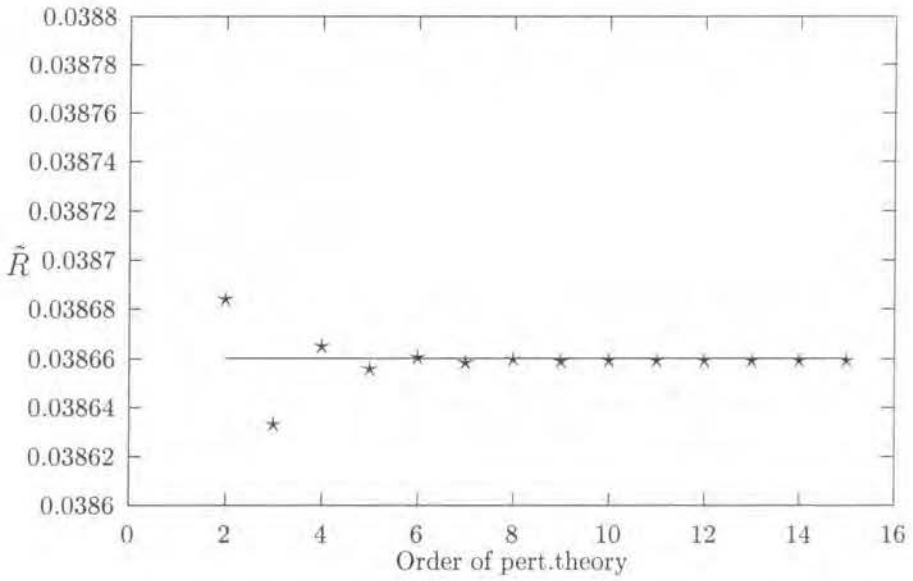

Figure 5.6: As Fig 5, but at $\sqrt{s} = M_Z$

measured with

$$\alpha^{-1} \equiv \alpha(0)^{-1} = 137.03599976(50) \,. \tag{5.51}$$

If one wishes to evolve α away from $s = 0$ to obtain $\alpha(s)$, however, then one needs to know leptonic and hadronic corrections,

$$\alpha(s)^{-1} = (1 - \Delta\alpha_{\rm lep}(s) - \Delta\alpha_{\rm had}(s) - \Delta\alpha_{\rm top}(s))\alpha^{-1} \,. \tag{5.52}$$

Whilst the leptonic corrections are known at three loops and are well-determined [92], the hadronic corrections for the contribution of the five lightest flavours, which we have denoted $\Delta\alpha_{\rm had}(s)$, is rather poorly determined and has to be reconstructed from the s-dependence of $\tilde{R}(s)$ using a dispersion relation. The contribution of the heaviest flavour $\Delta\alpha_{\rm top}(s)$ is rather well-determined and can be included separately. The value of $\alpha(M_Z^2)$ is of particular relevance since it limits the precision with which the unknown Higgs mass M_H of the Standard Model can be predicted from precision electroweak corrections [91]. Taking $s = M_Z^2$ we have [35] $\Delta\alpha_{\rm lep}(M_Z^2) = 314.98\times10^{-4}$ and $\Delta\alpha_{\rm top}(M_Z^2) = -0.76\times10^{-4}$. For the hadronic contribution we can use the dispersion relation,

$$\Delta\alpha_{\rm had}(M_Z^2) = -\frac{\alpha M_Z^2}{3\pi} PV \int_{4m_\pi^2}^{\infty} ds \frac{R(s)}{s(s - M_Z^2)} \,. \tag{5.53}$$

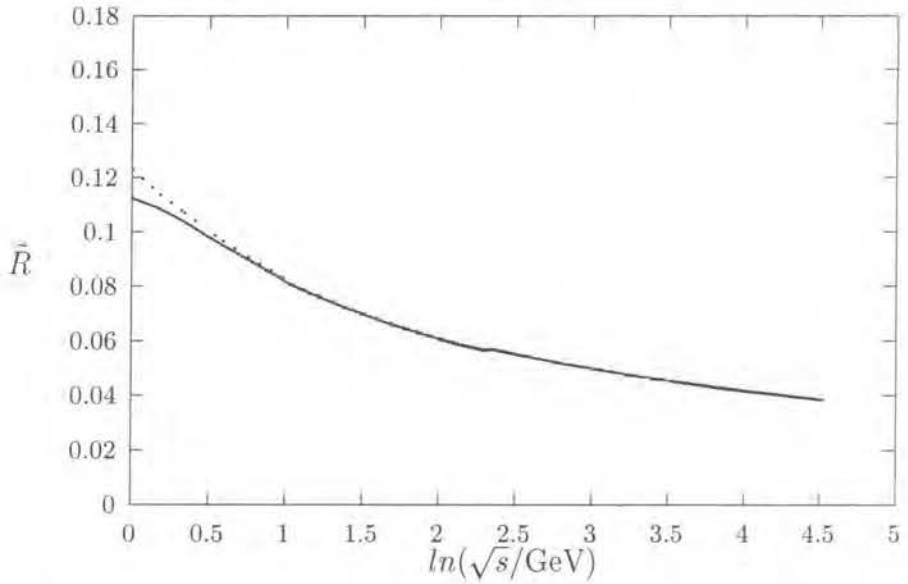

Figure 5.7: Fixed order (dashed line) and all-orders renormalon resummations (solid line) for $\tilde{R}(s)$ versus $ln(\sqrt{s}/\mathrm{GeV})$, over the range $1 < \sqrt{s} < 91$ GeV.

In Ref. [93] new exclusive data from BES-II [95] and Novosibirsk [94] have been used to extract $R(s)$ in the low energy region, with NNLO fixed order perturbative QCD used to evaluate it in the ranges $2.8 < \sqrt{s} < 3.74$ and $5 < \sqrt{s} < \infty$. We plan to approximate $R(s)$ in these latter ranges using the all-orders and NNLO fixed-order results for $\tilde{R}(s)$, as plotted in Figure 5.7. We shall use the exclusive data results as in Ref. [93], in the remaining energy ranges. Taking $\alpha_s(M_z^2) = 0.119$ we shall then determine $\alpha(M_Z^2)$ from the fixed-order CORGI results, and the all-orders leading-b resummed results. Since these results are contour-improved they include a resummation of analytical continuation terms not included in the fixed-order perturbative results used in [93]. We are interested in establishing if these terms and the uncalculated higher-order corrections, as estimated by the leading-b approximation, cause a significant shift in $\alpha(M_Z^2)$, and whether this has any ramifications for the constraints on M_H. In the region $2.8 < \sqrt{s} < 3.74$ GeV we obtain $\Delta\alpha_{\mathrm{had}}(M_Z^2) = (9.5424{\times}10^{-4},\ 9.7075{\times}10^{-4})$ for the (fixed-order, all-orders) CORGI results, and in the region $5 < \sqrt{s} < \infty$ we find $\Delta\alpha_{\mathrm{had}}(M_Z^2) = (170.788{\times}10^{-4},\ 170.635{\times}10^{-4})$. The total contribution of the remaining energy regions, using the

fits to exclusive data of [93] is $\Delta\alpha_{\rm had}(M_Z^2) = (94.12 \pm 1.76) \times 10^{-4}$. We find correspondingly using Eq.(5.52), $\alpha(M_Z^2)^{-1} = (128.967 \pm 0.024, 128.971 \pm 0.024)$, to be compared with $\alpha(M_Z^2)^{-1} = 128.978 \pm 0.027$ quoted in Ref [93]. In [93] there is the perturbative QCD error over two energy regions $2.8 < \sqrt{s} < 3.74$ and $5 < \sqrt{s} < \infty$, coming from varying m_c, m_b, M_z within the uncertainties in the [105], and varying the scale $\alpha_s(cs)$ in the range $0.25<c<4$, in our treatment we do not have this sort of error. Errors on the remaining energy regions are experimental errors which are quoted from [93]. We conclude that the analytical continuation terms and uncalculated higher order perturbative corrections do not cause a significant change in $\alpha(M_Z^2)$, and their inclusion does nothing to modify the conclusions of Ref [93].

5.6 All-orders CORGI resummations for the scalar correlator

The Higgs decay width to a quark anti-quark pair will be of fundamental phenomenological importance. In practice the decay to a $b\bar{b}$ will be the dominant contribution

$$\Gamma(H \to b\bar{b}) = \frac{3G_F}{4\sqrt{2}\pi} M_H m_b^2(M_H^2) \mathcal{R}(M_H^2) \,. \tag{5.54}$$

Here M_H is the Higgs mass and $m_b(M_H^2)$ is the running b-quark mass. $\mathcal{R}(M_H^2)$ is a coefficient function with a perturbative expansion

$$\mathcal{R}(M_H^2) = 1 + \sum_{n>0} r_n a^n \,, \tag{5.55}$$

where the coefficients r_1, r_2, r_3 have been exactly computed [96]. $\mathcal{R}$ can be straightforwardly related to the scalar correlator $\Pi_s(s)$. One can define an analogue of the vector Adler D-function so that

$$D(s) = s \frac{d}{ds} \left[\frac{\Pi_s(s)}{s} \right] \,. \tag{5.56}$$

This may be written in terms of the coefficient function $\mathcal{D}(s)$ where

$$D(s) = \frac{3}{8\pi^2} (m_b(s))^2 \mathcal{D}(s) \,, \tag{5.57}$$

and $\mathcal{D}$ has the perturbative expansion,

$$\mathcal{D}(s) = 1 + \sum_{n>0} d_n a^n. \tag{5.58}$$

$m_b^2(s)\mathcal{R}(s)$ can be related to $m_b^2(-s)\mathcal{D}(-s)$ by analytical continuation from Euclidean to Minkowski, and one can write a representation of the same form as Eq.(5.18) with

$$m_b^2(M_H^2)\,\mathcal{R}(M_H^2) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\; m_b^2(e^{i\theta}M_H^2)\mathcal{D}(e^{i\theta}M_H^2)\,. \tag{5.59}$$

To proceed further we can express the running mass in terms of an RG-invariant mass $\hat{m}_b$ and the mass anomalous dimension $\gamma_m(a)$, defined by

$$-\frac{d\ln(m(s))}{d\ln(s)} = \gamma_m(a) = \sum_{i\geq 0}\gamma_i a^{i+1}\,. \tag{5.60}$$

We can then write (c.f Eqs.(3.7) and (3.8))

$$m_b{}^2(s) = \hat{m}_b^2 b^{\frac{4\gamma_0}{b}}\left(\frac{a}{1+ca}\right)^{\frac{4\gamma_0}{b}} \exp\left[4\int_0^a dx \frac{\gamma_1 + (\gamma_1 c + \gamma_2 - \gamma_0 c_2)x + \ldots}{b(1+cx)(1+cx+c_2x^2+\ldots)}\right]\,. \tag{5.61}$$

The $b^{\frac{4\gamma_0}{b}}$ is the standard normalization of the definition of the RG-invariant mass $\hat{m}_b$. One can then write a CORGI series for $m_b^2(s)\ \mathcal{D}(s)$ exactly equivalent to that for the moments of structure functions in Eq.(3.26)

$$m_b^2(s)\mathcal{D}(s) = \hat{m}_b^2 b^{\frac{4\gamma_0}{b}}\left(\frac{a_0(s)}{1+ca_0(s)}\right)^{\frac{4\gamma_0}{b}}(1 + X_2 a_0^2(s) + X_3 a_0^3(s) + \ldots + X_n a_0^n(s) + \ldots)\,, \tag{5.62}$$

where $a_0(s)$ denotes the CORGI coupling which is again defined in terms of the Lambert W-function as in Eq.(2.83), and with the anomalous dimension present one now has $\Lambda_D = \exp[(\frac{\gamma_1}{\gamma_0 b}) + (\frac{d}{4\gamma_0})](\frac{2c}{b})^{-\frac{c}{b}}\Lambda_{\overline{MS}}$, with d the coefficient d_1 in the $\overline{MS}$ factorization and renormalization scheme with $M^2 = \mu^2 = s$ (M denoting the factorization scale) [76]. The exactly known CORGI invariants X_2 and X_3 follow from Eqs.(3.24), and allowing for the different definition of the anomalous dimension one needs to replace d_i by $4\gamma_i$. Lumping various inessential prefactors together we can define

$$\Gamma(H\to b\bar{b}) = \frac{3G_F}{4\sqrt{2}\pi} M_H \hat{m}_b^2 b^{4\gamma_0/b}\Gamma\,, \tag{5.63}$$

where Γ has the contour-improved CORGI representation,

$$\Gamma = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\left(\frac{\bar{a}_0}{1+c\bar{a}_0}\right)^{4\gamma_0/b}(1 + X_2\bar{a}_0^2 + X_3\bar{a}_0^3 + \sum_{n>3} X_n^{(L)}\bar{a}_0^n)\,, \tag{5.64}$$

with $\bar{a}_0 = a_0(e^{i\theta}M_H^2)$.

In the scalar case one will have coefficients with the structure

$$d_n = d_n^{[n-1]}N_f^{n-1} + d_{n-1}^{[n-2]}N_f^{n-2} + \ldots + d_n^{[0]}\,, \tag{5.65}$$

and after replacing $N_f = (\frac{33}{2} - 3b)$, as before, one arrives at a leading-b term with the structure $d_n^{(L)}{=}(-3)^{n-1}d_n^{[n-1]}b^{n-1}$, with one less power of b. The anomalous dimension coefficients will have the structure $\gamma_n^{(L)} = \gamma_n^{(n)}b^n$, but since the anomalous dimension $\gamma_m(a)$ does not contain renormalons there is no motivation for making this approximation, and it is poor in practice, as noted in Ref. [97]. Whilst an all-orders result for $\gamma_n^{(L)}$ does exist [97], we shall follow Ref. [97] and set $\gamma_n^{(L)} = 0$ for $n > 0$, retaining only γ_0. The all-orders result for $X_n^{(L)}$ follow straightforwardly from $d_n^{(L)}$. From the large-N_f results of Ref. [97] for the scalar correlator one can obtain an explicit all-orders expression for $d_n^{(L)}(V)$ (in the V-scheme) analogous to Eq.(5.26) in the vector case. For n even one has,

$$\begin{aligned} d_n^{(L)}(V) = &-\frac{32}{3}\frac{1}{2^{n+1}}\left(1-\frac{1}{2^n}\right)\zeta(n+1)n!b^{n-1} + \left(\frac{4}{n}+\frac{4}{3}\right)\left(\frac{1}{2}\right)^{n-1}n!b^{n-1} \\ &-\left(\frac{1}{n}+\frac{1}{3}\right)\left(\frac{1}{4}\right)^{n-1}n!b^{n-1}\,, \end{aligned} \tag{5.66}$$

whilst for odd n one has,

$$d_n^{(L)}(V) = \left(\frac{4}{n}+\frac{4}{3}\right)\left(\frac{1}{2}\right)^{n-1}n!b^{n-1} - \left(\frac{1}{n}+\frac{1}{3}\right)\left(\frac{1}{4}\right)^{n-1}n!b^{n-1}\,. \tag{5.67}$$

As in the vector case one can define a leading-b resummation

$$\mathcal{D}^{(L)} = 1 + \sum_{k=1}^{\infty} d_k^{(L)}(V)a^k\,, \tag{5.68}$$

analogous to Eq.(5.27), which may be defined as a regulated Borel sum

$$\mathcal{D}^{(L)}(1/a) = 1 + PV\int_0^{\infty} dz\, e^{-z/a}[G_-(z) + G_+(z)]\,. \tag{5.69}$$

Here $G_-(z)$ and $G_+(z)$ are the contributions to the Borel transform from UV and IR renormalons, respectively. One has (in the V-scheme) [97]

$$\begin{aligned} G_-(z) &= -\frac{4}{3}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2(1+bz/2k)^2} \\ G_+(z) &= \frac{4}{(1-bz/2)} - \frac{1}{(1-bz/4)} + \frac{4}{3}\sum_{k=3}^{\infty}\frac{(-1)^k}{k^2(1-bz/2k)^2} \end{aligned} \tag{5.70}$$

From these expressions one can read off the residues A_0, B_0 (cf. Eq.(5.29)), and one can then calculate $\mathcal{D}^{(L)}(F)|_{UV}$ and $\mathcal{D}^{(L)}(F)|_{IR}$ using Eqs.(5.32), (5.33). Finally

$$\mathcal{D}^{(L)}(F) = \mathcal{D}^{(L)}(F)|_{UV} + \mathcal{D}^{(L)}(F)|_{IR} \,, \tag{5.71}$$

with $F{\equiv}1/a_V$. To perform the leading-b CORGI resummation in Eq.(5.64) we simply need to relate $a_V^{(L)}$ and $a_0^{(L)}$ as we did in Section 5.3. In the presence of the anomalous dimension the RS-invariant combination ρ_0 in Eq.(5.37) is replaced by the factorization scheme and RS (FRS) invariant combination X_1 introduced in Eq.(3.22),

$$X_1 = 4\gamma_0 \ln\left(\frac{M}{\tilde{\Lambda}}\right) - \frac{4\gamma_1}{b} - d_1(M) \,, \tag{5.72}$$

where M is the factorization scale. Recalling that we have decided to set $\gamma_i^{(L)} = 0$ for $i > 0$ in our leading-b resummations , we can use Eq.(5.72) to relate $a_0^{(L)}$ and $a_V^{(L)}$,

$$\frac{1}{a_V^{(L)}} = \frac{1}{a_0^{(L)}} + \frac{b}{4\gamma_0} d_1^{(L)}(V) \,. \tag{5.73}$$

It then follows that the all-orders formal resummation in Eq.(5.64) is given by

$$\begin{aligned}\Gamma = &\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \left(\frac{\bar{a}_0}{1+c\bar{a}_0}\right)^{4\gamma_0/b} [1 - \frac{4\gamma_0}{b}\ln\left(1 + \frac{4\gamma_0}{b} d_1^{(L)}(V)\bar{a}_0\right) \\ &+\mathcal{D}^{(L)}\left(\frac{1}{\bar{a}_0} + \frac{b}{4\gamma_0} d_1^{(L)}(V)\right) + (X_2 - X_2^{(L)})\bar{a}_0^2 + (X_3 - X_3^{(L)})\bar{a}_0^3] \,. \end{aligned} \tag{5.74}$$

The logarithm term arises because of the fractional power $a^{4\gamma_0/b}$. Relating the V-scheme and CORGI couplings at the leading-b level one has,

$$a_V^{(L)\,4\gamma_0/b} = a_0^{(L)\,4\gamma_0/b}\left(1 + \frac{4\gamma_0}{b} d_1{}^{(L)}(V) a_0{}^{(L)}\right)^{-4\gamma_0/b} \,. \tag{5.75}$$

On expanding using the binomial theorem only the terms *linear* in γ_0 are leading in b, the remainder should be discarded. Writing the binomial expansion as $\exp[(-4\gamma_0/b)\ln S]$ $= 1 - (4\gamma_0/b)\ln S + O({\gamma_0}^2)$ the result follows. The same subtlety enters in deriving an analogue of Eq.(5.41) to generate the $X_n^{(L)}$ in terms of $d_n^{(L)}(V)$ explicitly given by Eqs.(5.66), (5.67). One finds

$$X_n^{(L)} = \mathcal{C}_n \left[\sum_{k=1}^{\infty} d_n^{(L)}(V)\left(\frac{a}{1 + (4\gamma_0/b) d_1^{(L)}(V) a}\right)^k - \frac{4\gamma_0}{b} ln\left(1 + \frac{4\gamma_0}{b} d_1{}^{(L)}(V) a\right)\right] . \tag{5.76}$$

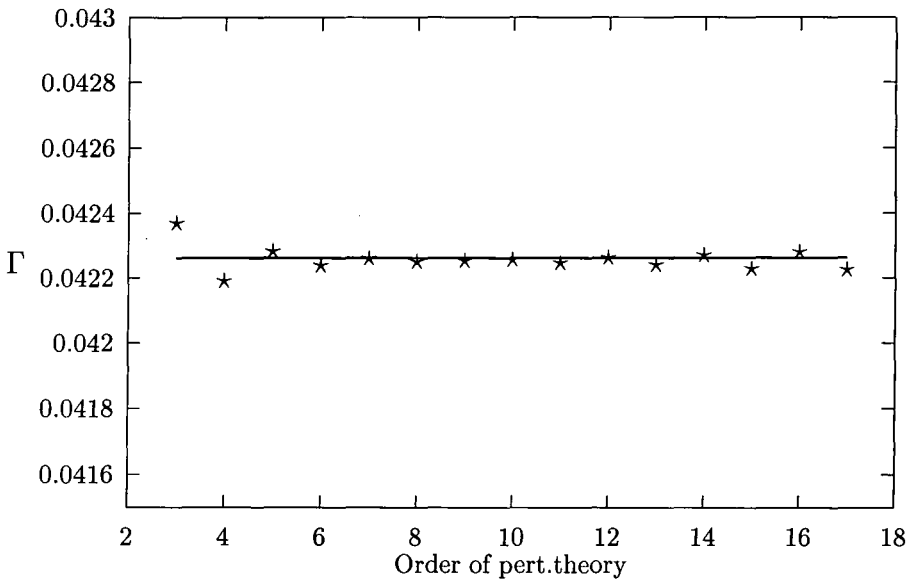

Figure 5.8: Fixed-order CORGI results for Γ ($M_H = 115$ GeV) in N^{n}LO perturbation theory (starred points), compared to the all-orders resummation (solid line)

These results can then be used to calculate all-orders and fixed-order CORGI predictions. We give in Figure 5.8 the analogue of Figs.1, 5, 6, for the Higgs decay width (with pre-factor set to unity) Γ, we set $M_H = 115$ GeV and $\alpha_s(M_Z^2) = 0.119$. The starred points show the fixed-order CORGI results, and the solid line the all-orders resummation. As before the agreement of the highest exactly calculated $n = 3$ fixed-order with the all-orders result is good. The fixed-order results track the resummed result up to $n = 12$, beyond which an oscillatory trend is noticeable. From Eq.(5.70) one can see that UV_1 and IR_1 renormalon singularities are present, and so the leading asymptotics are not dominated by UV_1 as in the vector case. The process of analytical continuation , however, serves to remove IR_1 [58,63] and so the leading asymptotics of Γ is expected to be dominated by the leading UV_1 renormalon, with resulting alternating-sign factorial behaviour. As discussed in Ref. [97] the presence of the leading IR_1 renormalon in $\mathcal{D}$ suggests that the obvious generalization of the Adler function in Eq.(5.56) may not be optimal, and an alternative is suggested. For our purposes here we are simply using D as a tool to compute the physical quantity Γ, and so this is not a problem. We should stress that the uncertainty in the prediction of the Higgs width, indicated

by comparing the fixed-order and all-orders results in Figure 5.8, will be much smaller than the error with which the width can be experimentally measured, and so our results will not effect the comparison of theory with experiment.

5.7 Analytic expressions for the CORGI contour improvement

In this section we wish to point out that we can obtain explicit analytic expressions for the CORGI fixed-order contour improved results for $\tilde{R}(s)$ in terms of the Lambert W-function, eliminating the need for numerical Simpson's rule evaluation. From Eq.(2.83) we see that we can write the CORGI coupling $\bar{a}_0 = a_0(e^{i\theta}s)$ in terms of the Lambert W-function as,

$$\bar{a}_0 = \frac{-1}{c[1 + W(Ae^{iK\theta})]} , \tag{5.77}$$

where

$$A(s) = \frac{-1}{e}\left(\frac{\sqrt{s}}{\Lambda_D}\right)^{-b/c} , K = \frac{-b}{2c} . \tag{5.78}$$

Thus after the contour integration the X_{n-1} coefficient multiplies

$$\begin{aligned} A_n(s) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \bar{a}_0^{-n} = \frac{1}{2\pi}\int_{-\pi}^{0} d\theta \frac{(-1)^n}{c^n}[1 + W_1(A(s)e^{iK\theta})]^{-n} \\ + \frac{1}{2\pi}\int_{0}^{\pi} d\theta \frac{(-1)^n}{c^n}[1 + W_{-1}(A(s)e^{iK\theta})]^{-n} , \end{aligned} \tag{5.79}$$

where the appropriate branches of the W-function are to be used in the two regions of integration. By making the change of variable $w = W(A(s)e^{iK\theta})$ we can then obtain the above integrals in the form,

$$\frac{(-1)^n}{2iKc^n\pi}\int \frac{dw}{w(1+w)^{n-1}} . \tag{5.80}$$

The w-integral is elementary, and including the limits of integration, and noting that $W_1(A(s)e^{-iK\pi}) = [W_{-1}(A(s)e^{iK\pi})]^*$, we obtain the explicit result,

$$A_n(s) = \frac{(-1)^n}{c^n K\pi} Im\left[\ln\left(\frac{W_{-1}(A(s)e^{iK\pi})}{1 + W_{-1}(A(s)e^{ik\pi})}\right) + \sum_{k=1}^{n-2}\frac{1}{k(1 + W_{-1}(A(s)e^{iK\pi}))^k}\right] , \tag{5.81}$$

for $n > 2$. For n=1 we have $A_1(s) = (-1/(\pi Kc))Im[\ln W_{-1}(A(s)e^{iK\pi})]$. We finally obtain the CORGI contour-improved fixed-order results in the form,

$$\tilde{R}(s) = A_1(s) + \sum_{k=2}^{\infty} X_k A_{k+1}(s) \,. \tag{5.82}$$

In the scalar correlator case analytic results can also be obtained. The X_n coefficient in the CORGI series for Γ in Eq.(5.64) will multiply

$$\begin{aligned} A_n(M_H^2) &\equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \left(\frac{\bar{a}_0}{1+c\bar{a}_0}\right)^{4\gamma_0/b} \bar{a}_0^n \\ &= \frac{1}{2\pi}\int_{-\pi}^{0} d\theta \frac{(-1)^n}{\left[-W_1(Ae^{iK\theta})\right]^{4\gamma_0/b}} \frac{c^{-n-4\gamma_0/b}}{\left[1+W_1(Ae^{iK\theta})\right]^n} \\ &+ \frac{1}{2\pi}\int_{0}^{\pi} d\theta \frac{(-1)^n}{\left[-W_{-1}(Ae^{iK\theta})\right]^{4\gamma_0/b}} \frac{c^{-n-4\gamma_0/b}}{\left[1+W_{-1}(Ae^{iK\theta})\right]^n} \,. \end{aligned} \tag{5.83}$$

Here $A = A(M_H^2)$. Making the change of variable $\omega = -W(Ae^{iK\theta})$ one can then obtain the above integrals in the form,

$$\frac{(-1)^n}{2\pi i K c^{n+4\gamma_0/b}} \int d\omega \frac{\omega^{(-4\gamma_0/b)-1}}{(1-\omega)^{n-1}} \,. \tag{5.84}$$

These integrals may be evaluated in terms of the Hypergeometric function $F(a,b;c;z)$ [98]. Inserting the limits of integration we obtain the explicit result

$$\begin{aligned} A_n(M_H^2) = \frac{-(-1)^n b}{4\pi K c^{n+4\gamma_0/b}\gamma_0} Im \Bigg\{ &\left[W_{-1}(Ae^{iK\pi})\right]^{-\frac{4\gamma_0}{b}} \\ &F\left(n-1, -\frac{4\gamma_0}{b}; 1-\frac{4\gamma_0}{b}; W_{-1}(Ae^{iK\pi})\right)\Bigg\} \,. \end{aligned} \tag{5.85}$$

We can finally write the CORGI contour-improved result in the form

$$\Gamma = A_0(M_H^2) + \sum_{k=2}^{\infty} X_k A_k(M_H^2) \,, \tag{5.86}$$

with

$$A_0(M_H^2) = \frac{1}{\pi K c^{4\gamma_0/b}} Im \left[\frac{b}{4\gamma_0}\left(W_{-1}(Ae^{iK\pi})\right)^{-4\gamma_0/b} + \frac{\left(W_{-1}(Ae^{iK\pi})\right)^{1-4\gamma_0/b}}{(1-\frac{4\gamma_0}{b})}\right] \,. \tag{5.87}$$

Chapter 6

Conclusions

The purpose of this thesis has been to investigate the way in which we can utilize perturbation theory to broaden our understanding of Quantum Chromodynamics (QCD). By increasing the accuracy and reliability of perturbative calculations we strive to reinforce our confidence in QCD as the fundamental quantum field theory of strong interactions.

This thesis covers two research subjects. The first one concerns the long standing problem of renormalization and factorization scheme dependence in QCD predictions. The renormalization scale, μ, dependence of fixed-order QCD perturbation theory is an artefact of the way renormalization group (RG) improvement is customarily performed. The two crucial features are the use of a scale μ proportional to the physical energy scale, Q, of the process, and the truncation of the perturbation series at fixed-order. As argued recently [76] one should rather keep μ *strictly independent* of Q. Fixed-order perturbation theory with μ constant does not then satisfy asymptotic freedom, and one is forced to sum to all-orders the RG-predictable unphysical logarithms of μ and physical ultraviolet (UV) logarithms of Q from which the perturbative coefficients are built. This so-called "Complete RG-improvement" (CORGI) [76] serves to cancel all μ-dependence between the unphysical renormalised coupling $\alpha_s(\mu)$, and the unphysical logarithms of μ in the coefficients, and one directly trades unphysical μ dependence for the physical Q-dependence. The CORGI approach as formulated in [76] is exactly

equivalent at NLO to the Effective Charge approach of Grunberg [12, 13] in which the UV logarithms are also completely resummed in exactly the same way, whilst the remaining RG-predictable effects are parametrized in a different, but *a priori* equally reasonable way. As we discussed in Chapter 3 it is possible to extend the ideology of CORGI to the case where two unphysical scales enter. We studied the example of the arbitrary factorization scale dependence of moments of structure functions in deep-inelastic leptoproduction. Here an anologous situation holds, where in addition to μ and c_n renormalization scheme parameters, there will be a set of factorisation scheme parameters, M and d_n. Once again , a resummation of the ultraviolet logarithms results in the dependence on these unphysical scales disappearing.

We showed that the CORGI formalism was equivalent to directly relating observables to the dimensional transmutation parameter, which arises as a constant of integration when the energy dependence imposed by dimensional analysis in integrand. This integration automatically resums altraviolet logarithms to all-orders, and the renormalization scale μ cancels and, is seen to be a completely redundant parameter with no physical significance.

In the other part of the thesis, we focussed on obtaining exact fixed-order, and leading-b estimated all-orders results for various inclusive QCD Minkowski observables , related to the vector correlator. These could be expressed as a contour integral of the suitably weighted Euclidean Adler $D(-s)$-function in the complex energy squared plane. $D(s)$ is truncated at some fixed-order and the integral performed numerically as described in Section 5.2. In this way contour-improved predictions are obtained, in which an infinite subset of known and potentially large analytical continuation terms are resummed to all-orders. By employing the CORGI approach, as discussed in Section 5.3, we could further resum to all-orders the complete set of ultraviolet logarithms involving s, which build the s-dependence of $D(s)$, avoiding any dependence on an arbitrary renormalization scale μ. The remaining approximation is the missing higher-order CORGI invariants X_i $(i > 2)$, which remain unknown since the perturbative coefficients

have only been calculated to NNLO so far. We used the so-called leading-b approximation to estimate these. Using the exact large-N_f all-orders results for $D(s)$ [54] we were able to sum the CORGI series to all-orders in terms of a sum of exponential integral functions, corresponding to the contributions of the ultraviolet and infrared renormalons in the Borel plane. This was technically far more straightforward than previous analogous resummations of the Effective Charge beta-function for $D(s)$ [74].

The renormalon singularities are divided into UV and IR singularities. In QCD, due to asymptotic freedom and the resulting sign of the first coefficient of the β-function, the UV renormalons lie on the negative real axis of the Borel variable and pose no problems in the reconstruction of a physical quantity from its divergent perturbative expansion. The IR renormalons, however, are of more interest since they introduce a genuine ambiguity into the reconstruction process. Despite the hindrance to the resummation programme caused by IR renormalons, we are obliged to accept that they are necessary for QCD to be consistent. That is, we believe that IR renormalons are directly connected to non-perturbative effects characterised by the operator product expansion (OPE) and that the only way in which one can unite perturbation theory with the non-perturbative regime is by systematic cancellation between IR renormalon induced ambiguities and ambiguities originating in the OPE. It was in this light that we focused on the Adler D-function where we demonstrated, by using exact large-N_f results from QED, that the singularity structure in the Borel plane was precisely in line with expectations from OPE. Most notably this involved the absence of the first IR renormalon at $z = \frac{2}{b}$, corresponding to the absence from the OPE of an operator with dimension two.

By comparing the NNLO fixed-order CORGI results to the all-orders resummations we estimated the uncertainty in $\alpha_s(M_Z^2)$, extracted from experimental measurements of R_τ, to be $\delta\alpha_s(M_Z^2)\approx 0.003$. Measurments of $R(s)$ at low energies are crucial for determining $\alpha(M_Z^2)$, for the evolved QED fine structure constant. Accurate knowledge of the value of this quantity is vital in constraining the mass of the Higgs from electroweak

fits to LEP data. We showed that using all-orders and fixed-order contour-improved CORGI results for $R(s)$ gave results for the hadronic corrections to the QED coupling $\alpha(M_Z^2)$ which did not differ significantly from those obtained using conventional fixed-order perturbation theory in Ref. [93]. Finally we showed how recent all-orders large-N_f results for the scalar correlator [97] could be used to perform analogous resummations for the Higgs decay width to a heavy quark pair. We finally noted that the CORGI contour-improvement for the R-ratio can be written in analytic form in terms of the Lambert W-function, and for the Higgs width in terms of the Hypergeometric function and Lambert W-function, thus avoiding the need to use a numerical Simpson's Rule evaluation.

There are many possible investigations to be pursued using these methods. In particular it would be interesting to use the resummed results for $R(s)$ in lower energy ranges where conventionally inclusive or exclusive data has been used. The scalar correlator results could also be used to investigate more carefully the uncertainties in estimates of the strange quark mass using Sum Rule techniques.

Appendix A

Analytic expression for structure function moments valid up to a^4

In addition to the expansion form (3.10) for the structure function moments $\mathcal{M}$ we can obtain a closed analytical form for $\mathcal{M}$ valid up to a^4, which has some advantages with repect the expanded form. First if we put in this closed form $c_2 = 0$, $c_3 = 0$, we obtain the complete analytic form for the structure function moments as in Ref. [25], and secondly this analytical form is extendable to the more general cases where we have d_4, d_5,... in the numenator and c_4,c_5,... in the denominator of first integrand of (3.5) which makes the comparison of the form of the structure function moments in different orders more straightforward.

To get this closed analytical form for $\mathcal{M}$ we take the following steps. First recall that we defined the structure function moments as

$$\mathcal{M}_n(Q) =< \mathcal{O}_n(M) > \mathcal{C}_n(Q, a(\mu), \mu, M) . \tag{A.1}$$

As in Section 3.2, we use $\tilde{a}$ to stand for $a(\mu)$. The Coefficient function up to $\tilde{a}^4$ is

$$\mathcal{C}_4 = 1 + r_1\tilde{a} + r_2\tilde{a}^2 + r_3\tilde{a}^3 . \tag{A.2}$$

The operator matrix element was defined as

$$\mathcal{O}(M) >= A \exp[\int_0^a \frac{\gamma(x)}{\beta(x)} dx - \int_0^\infty \frac{\gamma^{(1)}(x)}{\beta^{(2)}(x)} dx] . \tag{A.3}$$

and the anolmalous dimention to fourth order is

$$\gamma(x) = -dx - d_1x^2 - d_2x^3 - d_3x^4 \,, \tag{A.4}$$

and the truncated β function is

$$\beta(x) = -bx^2(1 + cx + c_2x^2 + c_3x^3) \,, \tag{A.5}$$

where

$$\gamma^{(1)}(x) = -dx, \quad \beta^{(2)}(x) = -bx^2(1 + cx) \,. \tag{A.6}$$

Consequently the first integral in Eq.(3.5) will appear as

$$\int_0^a \frac{\gamma(x)}{\beta(x)}\,dx = \int_0^a \frac{dx + d_1x^2 + d_2x^3 + d_3x^4}{bx^2(1 + cx + c_2x^2 + c_3x^3)}\,dx \,. \tag{A.7}$$

Expanding the denominator in Eq.(A.7) yields

$$\begin{aligned}
&(1 + cx + c_2x^2 + c_3x^3)^{-1} = [(1 + cx) + (c_2x^2 + c_3x^3)]^{-1} \\
&= (1 + cx)^{-1} - (1 + cx)^{-2}(c_2x^2 + c_3x^3) + ... \\
&= \frac{1}{1 + cx} - \frac{c_2x^2}{(1 + cx)^2} - \frac{c_3x^3}{(1 + cx)^2} + ... \,,
\end{aligned} \tag{A.8}$$

here we kept only terms which are linear with repect to c_2 and c_3. Other expansion terms are of order a^5 and higher, which are not our concern therefore. We then obtain

$$\begin{aligned}
&\int_0^a \frac{\gamma(x)}{\beta(x)}\,dx = \frac{d}{b}\int_0^a \frac{1}{x(1 + cx + c_2x^2 + c_3x^3)}\,dx \\
&+\frac{d_1}{b}\int_0^a \frac{1}{(1 + cx + c_2x^2 + c_3x^3)}\,dx + \frac{d_2}{b}\int_0^a \frac{x}{(1 + cx + c_2x^2 + c_3x^3)}\,dx \\
&+\frac{d_3}{b}\int_0^a \frac{x^2}{(1 + cx + c_2x^2 + c_3x^3)}\,dx + \frac{d_3}{b}\int_0^a \frac{x}{(1 + cx + c_2x^2 + c_3x^3)}\,dx \,.
\end{aligned} \tag{A.9}$$

From Eq.(A.7) and Eq.(A.8) we get

$$\begin{aligned}
&\int_0^a \frac{\gamma(x)}{\beta(x)}\,dx = \frac{d}{b}[\int_0^a \frac{1}{x(1 + cx)}\,dx - c_2\int_0^a \frac{x}{x(1 + cx)^2}\,dx - c_3\int_0^a \frac{x^2}{x(1 + cx)^2}\,dx] \\
&+\frac{d_1}{b}[\int_0^a \frac{1}{(1 + cx)}\,dx - c_2\int_0^a \frac{x^2}{(1 + cx)^2}\,dx - c_3\int_0^a \frac{x^3}{(1 + cx)^2}\,dx] \\
&+\frac{d_2}{b}[\int_0^a \frac{x}{(1 + cx)}\,dx - c_2\int_0^a \frac{x^3}{(1 + cx)^2}\,dx - c_3\int_0^a \frac{x^4}{(1 + cx)^2}\,dx] \\
&+\frac{d_3}{b}[\int_0^a \frac{x^2}{(1 + cx)}\,dx - c_2\int_0^a \frac{x^4}{(1 + cx)^2}\,dx - c_3\int_0^a \frac{x^5}{(1 + cx)^2}\,dx] \,,
\end{aligned} \tag{A.10}$$

and from Eq.(A.6)

$$\int_0^\infty \frac{\gamma^{(1)}(x)}{\beta^{(2)}(x)}\,dx = \int_0^\infty \frac{dx}{bx^2(1+cx)}\,dx\ . \tag{A.11}$$

Substituting Eq.(A.10) and Eq.(A.11) in Eq.(A.3) and using the obtained results in Eq.(A.1), after simplification we obtain the following analytical result for $\mathcal{M}$

$$\begin{aligned}
\mathcal{M} = {} & A(ca)^{\frac{d}{b}}(1+ca)^{\frac{-d}{b}+\frac{d_1}{bc}-\frac{d_2}{bc^2}+\frac{d_3}{bc^3}+\frac{c_2}{b}(-\frac{d}{c^2}+2\frac{d_1}{c^3}-3\frac{d_2}{c^4}+4\frac{d_3}{c^5})-c_3(-2\frac{d}{bc^3}+3\frac{d_1}{bc^4}-4\frac{d_2}{bc^5}+5\frac{d_3}{bc^6})} \\
& \exp(\frac{ad_2}{bc})\exp(\frac{a(a^2c^2-2-ca)d_3}{2bc^2(1+ca)})\exp[c_2(\frac{ad}{bc(1+ca)}-\frac{d_1(ca^2+2a)}{bc^2(1+ca)} \\
& +\frac{d_2(-c^3a^3+3a^2c^2+6ca)}{2c^4(1+ca)b}-\frac{d_3(12a+6ca^2-2c^2a^3+c^3a^4)}{3c^4(1+ca)b})]\times \\
& \exp[c_3(\frac{-d(ca^2+2a)}{bc^2(1+ca)}+\frac{d_1(-c^3a^3+3a^2c^2+6ca)}{2c^4(1+ca)b)} \\
& -\frac{d_2(12a+6ca^2-2c^2a^3+c^3a^4)}{3c^4(1+ca)b} \\
& +\frac{d_3(60a-3c^4a^5+30ca^2-10c^2a^3+5c^3a^4)}{12bc^5(1+ca)})](1+r_1\tilde{a}+r_2\tilde{a}^2+r_3\tilde{a}^3)\ .
\end{aligned} \tag{A.12}$$

Appendix B

Structure function moments and partial derivatives

We can use the expansion form in Eq.(3.10) or the closed analytical form in Eq.(A.12) to obtain the partial derivaties of $\mathcal{M}$ with repect to seven unphysical quentites μ, M, d_1, d_2, d_3, c_2, c_3. In the following we use the closed analytical form to obtain these partial dereivaties.

To calculate partial derivatives of $\mathcal{M}$ with respect to μ, and to get the obtained results for r_1,r_2 and r_3 we begin with the chain rule as

$$\mu\frac{\partial \mathcal{M}}{\partial \mu} = \mu\frac{\partial \mathcal{M}}{\partial \tilde{a}}\frac{\partial \tilde{a}}{\partial \mu} + \mu\frac{\partial \mathcal{M}}{\partial r_1}\frac{\partial r_1}{\partial \mu} + \mu\frac{\partial \mathcal{M}}{\partial r_2}\frac{\partial r_2}{\partial \mu} + \mu\frac{\partial \mathcal{M}}{\partial r_3}\frac{\partial r_3}{\partial \mu}\,, \tag{B.1}$$

where

$$\mathcal{M} = < \mathcal{O}(M) > (1 + r_1\tilde{a} + r_2\tilde{a}^2 + r_3\tilde{a}^3)\,, \tag{B.2}$$

consequently

$$\begin{aligned}\mu\frac{\partial \mathcal{M}}{\partial \mu} &= \frac{\partial \mathcal{M}}{\partial \tilde{a}}(\mu\frac{\partial \tilde{a}}{\partial \mu}) + < \mathcal{O} > \tilde{a}(\mu\frac{\partial r_1}{\partial \mu}) \\ &+ < \mathcal{O} > \tilde{a}^2(\mu\frac{\partial r_2}{\partial \mu}) + < \mathcal{O} > \tilde{a}^3(\mu\frac{\partial r_3}{\partial \mu})\,.\end{aligned} \tag{B.3}$$

From Eq.(B.2)

$$\frac{\partial \mathcal{M}}{\partial \tilde{a}} = (r_1 + 2r_2\tilde{a} + 3r_3\tilde{a}^2) < \mathcal{O} >\,, \tag{B.4}$$

and we know

$$\mu\frac{\partial\tilde{a}}{\partial\mu} = \beta(\tilde{a}) = -b\tilde{a}^2(1 + c\tilde{a} + c_2\tilde{a}^2 + c_3\tilde{a}^3) \,. \tag{B.5}$$

Substituting Eq.(B.4) and Eq.(B.5) into Eq.(B.1) yields

$$\mu\frac{\partial\mathcal{M}}{\partial\mu} = \mathcal{M}\frac{r_1 + 2r_2\tilde{a} + 3r_3\tilde{a}^2}{1 + r_1\tilde{a} + r_2\tilde{a}^2 + r_3\tilde{a}^3}(-b\tilde{a}^2(1 + c\tilde{a} + c_2\tilde{a}^2 + c_3\tilde{a}^3)) \tag{B.6}$$

$$+\mathcal{M}\frac{\tilde{a}(\mu\frac{\partial r_1}{\partial\mu}) + \tilde{a}^2(\mu\frac{\partial r_2}{\partial\mu}) + \tilde{a}^3(\mu\frac{\partial r_3}{\partial\mu})}{1 + r_1\tilde{a} + r_2\tilde{a}^2 + r_3\tilde{a}^3} \,, \tag{B.7}$$

equivalently we can consider

$$\frac{\mathcal{M}}{1 + r_1\tilde{a} + r_2\tilde{a}^2 + r_3\tilde{a}^3}[-(r_1 + 2r_2\tilde{a} + 3r_3\tilde{a}^2)b\tilde{a}^2(1 + c\tilde{a} + c_2\tilde{a}^2 + c_3\tilde{a}^3)$$
$$+\tilde{a}(\mu\frac{\partial r_1}{\partial\mu}) + \tilde{a}^2(\mu\frac{\partial r_2}{\partial\mu}) + \tilde{a}^3(\mu\frac{\partial r_3}{\partial\mu})] \,, \tag{B.8}$$

and from there, self-consistency principle (i.e vanishing of $O(a)$, $O(a^2)$ and $O(a^3)$ terms) demands

$$\mu\frac{\partial r_1}{\partial\mu} = 0, \quad \mu\frac{\partial r_2}{\partial\mu} = r_1 b, \quad \mu\frac{\partial r_3}{\partial\mu} = 2r_2 b + r_1 bc \,. \tag{B.9}$$

To calculate the partial derivative of $\mathcal{M}$ with respect to M and obtain results for partial derivatives of r_1, r_2,..., using the chain rule yields

$$M\frac{\partial\mathcal{M}}{\partial M} = M\frac{\partial\mathcal{M}}{\partial< \mathcal{O} >}\frac{\partial< \mathcal{O} >}{\partial M} + M\frac{\partial\mathcal{M}}{\partial r_1}\frac{\partial r_1}{\partial M} + M\frac{\partial\mathcal{M}}{\partial r_2}\frac{\partial r_2}{\partial M}$$
$$+M\frac{\partial\mathcal{M}}{\partial r_3}\frac{\partial r_3}{\partial M} \,. \tag{B.10}$$

Using Eq.(B.2) we get

$$\frac{\partial\mathcal{M}}{\partial< \mathcal{O} >}(M\frac{\partial< \mathcal{O} >}{\partial M}) + \mathcal{M}\frac{\tilde{a}(M\frac{\partial r_1}{\partial M}) + \tilde{a}^2(M\frac{\partial r_2}{\partial M}) + \tilde{a}^3(M\frac{\partial r_3}{\partial M})}{1 + r_1\tilde{a} + r_2\tilde{a}^2 + r_3\tilde{a}^3} \,, \tag{B.11}$$

on the other hand using the definition of anomalous dimension $\gamma_{\mathcal{O}}$ (c.f. Eq.(3.3)), we have

$$\frac{\partial\mathcal{M}}{\partial< \mathcal{O} >}(M\frac{\partial< \mathcal{O} >}{\partial M}) =< \mathcal{O} > \frac{\partial\mathcal{M}}{\partial< \mathcal{O} >}[(M\frac{\partial< \mathcal{O} >}{\partial M})\frac{1}{< \mathcal{O} >}]$$
$$= (< \mathcal{O} > C_4)\gamma_{\mathcal{O}}(a) = \mathcal{M}[-da - d_1a^2 - d_2a^3 - d_3a^4] \,. \tag{B.12}$$

Combining Eq.(B.11) and Eq.(B.12) yields

$$M\frac{\partial \mathcal{M}}{\partial M} = \mathcal{M}[(-da - d_1a^2 - d_2a^3 - d_3a^4) + \mathcal{M}\frac{\tilde{a}(M\frac{\partial r_1}{\partial M}) + \tilde{a}^2(M\frac{\partial r_2}{\partial M}) + \tilde{a}^3(M\frac{\partial r_3}{\partial M})}{1 + r_1\tilde{a} + r_2\tilde{a}^2 + r_3\tilde{a}^3}] . \quad \text{(B.13)}$$

By substituting the relation of the couplings with the two different scales,

$$a = \tilde{a} + b\tilde{a}^2 log\frac{M}{\mu} + b^2\tilde{a}^3 log^2\frac{M}{\mu} , \quad \text{(B.14)}$$

we obtain the following results

$$M\frac{\partial r_1}{\partial M} = d, \quad M\frac{\partial r_2}{\partial M} = d_1 + dr_1 - dblog\frac{M}{\mu} ,$$
$$M\frac{\partial r_3}{\partial M} = d_2 + d_1r_1 + dr_2 - dbr_1 log\frac{M}{\mu} - 2d_1blog\frac{M}{\mu} - db^2log^2\frac{M}{\mu} . \quad \text{(B.15)}$$

For the partial derivative of $\mathcal{M}$ with respect to d_1 and the related results we have to compute

$$\frac{\partial \mathcal{M}}{\partial d_1} , \quad \text{(B.16)}$$

by using chain rule where

$$\mathcal{M} = (ca)^{\frac{d}{b}}(1 + ca)^A e^E e^G e^B e^D (1 + r_1\tilde{a} + r_2\tilde{a}^2 + r_3\tilde{a}^3) . \quad \text{(B.17)}$$

A,B,D,E and G are defined from (A.12) as

$$A = \frac{-d}{b} + \frac{d_1}{bc} - \frac{d_2}{bc^2} + \frac{d_3}{bc^3} + \frac{c_2}{b}(\frac{d}{c^2} - 2\frac{d_1}{c^3} + 3\frac{d_2}{c^4} - 4\frac{d_3}{c^5}) + c_3(2\frac{d}{bc^3} - 3\frac{d_1}{bc^4} + 4\frac{d_2}{bc^5} - 5\frac{d_3}{bc^6}) \quad \text{(B.18)}$$

$$B = c_2[\frac{ad}{bc(1+ca)} - \frac{d_1(ca^2+2a)}{bc^2(1+ca)} + \frac{d_2(-c^3a^3+3a^2c^2+6ca)}{2c^4(1+ca)b} - \frac{d_3(12a+6ca^2-2c^2a^3+c^3a^4)}{3c^4(1+ca)b}] \quad \text{(B.19)}$$

$$D = c_3[\frac{-d(ca^2+2a)}{bc^2(1+ca)} + \frac{d_1(-c^3a^3+3a^2c^2+6ca)}{c^4(1+ca)b} - \frac{d_2(12a+6ca^2-2c^2a^3+c^3a^4)}{c^4(1+ca)b} + \frac{d_3(60a-3c^4a^5+30ca^2-10c^2a^3+5c^3a^4)}{12bc^5(1+ca)}] \quad \text{(B.20)}$$

$$E = \frac{ad_2}{bc}, G = \frac{a(a^2c^2-2-ca)d_3}{2bc^2(1+ca)} . \quad \text{(B.21)}$$

Using the chain rule, the partial derivative of M with repect to d_1 can be written as

$$\frac{\partial \mathcal{M}}{\partial d_1} = \frac{\partial \mathcal{M}}{\partial A}\frac{\partial A}{\partial d_1} + \frac{\partial \mathcal{M}}{\partial B}\frac{\partial B}{\partial d_1} + \frac{\partial \mathcal{M}}{\partial D}\frac{\partial D}{\partial d_1} + \frac{\partial \mathcal{M}}{\partial r_1}\frac{\partial r_1}{\partial d_1} + \frac{\partial \mathcal{M}}{\partial r_2}\frac{\partial r_2}{\partial d_1} + \frac{\partial \mathcal{M}}{\partial r_3}\frac{\partial r_3}{\partial d_1} . \quad \text{(B.22)}$$

From the definition of A,B and D we will get

$$\frac{\partial \mathcal{M}}{\partial A} = log(1+ca)\mathcal{M}, \quad \frac{\partial A}{\partial d_1} = \frac{1}{bc} - \frac{2c_2}{bc^3} + \frac{3c_3}{bc^4} , \quad \text{(B.23)}$$

$$\frac{\partial \mathcal{M}}{\partial B} = \mathcal{M}, \quad \frac{\partial B}{\partial d_1} = \frac{-c_2(ca^2+2a)}{bc^2(1+ca)} , \quad \text{(B.24)}$$

$$\frac{\partial \mathcal{M}}{\partial D} = \mathcal{M}, \quad \frac{\partial D}{\partial d_1} = \frac{c_3}{2}\frac{(c^3a^3 - 3a^2c^2 - 6ca)}{c^4(1+ca)b} , \quad \text{(B.25)}$$

substituting Eq.(B.23), Eq.(B.24) and Eq.(B.25) into Eq.(B.22) yields

$$\frac{\partial \mathcal{M}}{\partial d_1} = \mathcal{M}[log(1+ca)(\frac{1}{bc} - \frac{2c_2}{bc^3} + \frac{3c_3}{bc^4}) + \frac{c_2(ca^2+2a)}{bc^2(1+ca)}$$
$$+\frac{c_3}{2}\frac{(c^3a^3 - 3a^2c^2 - 6ca)}{c^4(1+ca)b)} + \frac{\tilde{a}(\frac{\partial r_1}{\partial d_1}) + \tilde{a}^2(\frac{\partial r_2}{\partial d_1}) + \tilde{a}^3(\frac{\partial r_3}{\partial d_1})}{1 + r_1\tilde{a} + r_2\tilde{a}^2 + r_3\tilde{a}^3}] . \quad \text{(B.26)}$$

Sustituting Eq.(B.14) in Eq.(B.26) and demanding self-consistency principle will result in

$$\frac{\partial r_1}{\partial d_1} = \frac{-1}{b}, \quad \frac{\partial r_2}{\partial d_1} = \frac{c}{2b} - log\frac{M}{\mu} - \frac{r_1}{b} ,$$
$$\frac{\partial r_3}{\partial d_1} = \frac{cr_1}{2b} - \frac{c^2}{3b} + (c - r_1)log\frac{M}{\mu} - \frac{r_2}{b} + \frac{c_2}{3b} - blog^2\frac{M}{\mu} . \quad \text{(B.27)}$$

For computing the partial derivative of $\mathcal{M}$ with repect to d_2 and the related results in a similar way we have

$$\frac{\partial \mathcal{M}}{\partial d_2} = \frac{\partial \mathcal{M}}{\partial A}\frac{\partial A}{\partial d_2} + \frac{\partial \mathcal{M}}{\partial E}\frac{\partial E}{\partial d_2} + \frac{\partial \mathcal{M}}{\partial B}\frac{\partial B}{\partial d_2} + \frac{\partial \mathcal{M}}{\partial D}\frac{\partial D}{\partial d_2} + \frac{\partial \mathcal{M}}{\partial r_1}\frac{\partial r_1}{\partial d_2} + \frac{\partial \mathcal{M}}{\partial r_2}\frac{\partial r_2}{\partial d_2} + \frac{\partial \mathcal{M}}{\partial r_3}\frac{\partial r_3}{\partial d_2} , \quad \text{(B.28)}$$

A,B,D,E are the same as before and we will get

$$\frac{\partial \mathcal{M}}{\partial d_2} = \mathcal{M}[log(1+ca)(\frac{-1}{bc^2} + \frac{3c_2}{bc^4} - \frac{4c_3}{bc^5}) + \frac{a}{bc} + \frac{c_2(-c^3a^3 + 3a^2c^2 + 6ca)}{2bc^4(1+ca)}$$
$$-c_3\frac{(c^3a^4 - 2a^3c^2 + 6ca^2 + 12a)}{c^4(1+ca)b} + \frac{\tilde{a}(\frac{\partial r_1}{\partial d_2}) + \tilde{a}^2(\frac{\partial r_2}{\partial d_2}) + \tilde{a}^3(\frac{\partial r_3}{\partial d_2})}{1 + r_1\tilde{a} + r_2\tilde{a}^2 + r_3\tilde{a}^3}] . \quad \text{(B.29)}$$

Form Eq.(B.29) we can obtain the partial derivatives of r_1, r_2 and r_3 with respect to d_2 and before that we need to substitute Eq.(B.14) in Eq.(B.29), then we will get

$$\frac{\partial r_1}{\partial d_2} = 0, \quad \frac{\partial r_2}{\partial d_2} = \frac{-1}{2b}, \quad \frac{\partial r_3}{\partial d_2} = \frac{c}{3b} - log\frac{\mu}{M} - \frac{r_1}{2b} . \quad \text{(B.30)}$$

In a similar way for $\frac{\partial \mathcal{M}}{\partial d_3}$ we will have

$$\frac{\partial \mathcal{M}}{\partial d_3} = \frac{\partial \mathcal{M}}{\partial A}\frac{\partial A}{\partial d_3} + \frac{\partial \mathcal{M}}{\partial G}\frac{\partial G}{\partial d_3} + \frac{\partial \mathcal{M}}{\partial B}\frac{\partial B}{\partial d_3} + \frac{\partial \mathcal{M}}{\partial D}\frac{\partial D}{\partial d_3} + \frac{\partial \mathcal{M}}{\partial r_1}\frac{\partial r_1}{\partial d_3} + \frac{\partial \mathcal{M}}{\partial r_2}\frac{\partial r_2}{\partial d_3} + \frac{\partial \mathcal{M}}{\partial r_3}\frac{\partial r_3}{\partial d_3}\,, \tag{B.31}$$

and from this

$$\begin{aligned}\frac{\partial \mathcal{M}}{\partial d_3} &= \mathcal{M}[log(1+ca)(\frac{1}{bc^3} - \frac{4c_2}{bc^5} + \frac{5c_3}{bc^6}) + \frac{a(a^2c^2 - 2 - ca)}{2b(1+ca)} \\ &- \frac{c_2(c^3a^3 - 2a^3c^2 + 6ca^2 + 12a)}{3bc^4(1+ca)} + c_3\frac{(-3c^4a^5 + 5c^3a^4 - 10a^3c^2 + 30c^2a + 60a)}{12c^5(1+ca)b} \\ &+ \frac{\tilde{a}(\frac{\partial r_1}{\partial d_3}) + \tilde{a}^2(\frac{\partial r_2}{\partial d_3}) + \tilde{a}^3(\frac{\partial r_3}{\partial d_3})}{1 + r_1\tilde{a} + r_2\tilde{a}^2 + r_3\tilde{a}^3}]\,,\end{aligned} \tag{B.32}$$

and substituting a by $\tilde{a}$ yields

$$\frac{\partial r_1}{\partial d_3} = 0, \quad \frac{\partial r_2}{\partial d_3} = 0, \quad \frac{\partial r_3}{\partial d_3} = \frac{-1}{3b}\,. \tag{B.33}$$

For the derivative of $\mathcal{M}$ with respect to c_2 we will get

$$\begin{aligned}\frac{\partial \mathcal{M}}{\partial c_2} &= \frac{\partial \mathcal{M}}{\partial A}\frac{\partial A}{\partial c_2} + \frac{\partial \mathcal{M}}{\partial B}\frac{\partial B}{\partial c_2} + \frac{\partial \mathcal{M}}{\partial a}\frac{\partial a}{\partial c_2} + \frac{\partial \mathcal{M}}{\partial \tilde{a}}\frac{\partial \tilde{a}}{\partial c_2} + \frac{\partial \mathcal{M}}{\partial r_1}\frac{\partial r_1}{\partial c_2} + \frac{\partial \mathcal{M}}{\partial r_2}\frac{\partial r_2}{\partial c_2} \\ &+ \frac{\partial \mathcal{M}}{\partial r_3}\frac{\partial r_3}{\partial c_2}\,,\end{aligned} \tag{B.34}$$

where A and B are the same as before.

Note that in the above we need the dependence of a and $\tilde{a}$ on c_2 in the form $\frac{\partial \mathcal{M}}{\partial a}\frac{\partial a}{\partial c_2}$ and $\frac{\partial \mathcal{M}}{\partial \tilde{a}}\frac{\partial \tilde{a}}{\partial c_2}$ and these dependences can be obtained from the Eq.(2.73) and the similar relation for $\tilde{a}$ (c.f [9]).

After calculating and substituting the partial derivatives in Eq.(B.34) we will get

$$\begin{aligned}\frac{\partial \mathcal{M}}{\partial c_2} &= \mathcal{M}[log(1+ca)(\frac{d}{bc^2} - \frac{2d_1}{bc^3} + \frac{3d_2}{bc^4} - \frac{4d_3}{c^5}) + \frac{ad}{cb(1+ca)} \\ &- \frac{d_1(ca^2 + 12a)}{bc^2(1+ca)} + \frac{d_2(-c^2a^3 + 3ca^2 + 60a)}{2bc^3(1+ca)} \\ &- d_3\frac{(3c^3a^4 - 2c^2a^3 + 6ca^2 + 12a)}{3c^4(1+ca)b} - \frac{d}{b}a^2 + \frac{dc}{b}a^3 - \frac{d_1}{b}a^3 - r_1\tilde{a}^3 \\ &+ \frac{\tilde{a}(\frac{\partial r_1}{\partial c_2}) + \tilde{a}^2(\frac{\partial r_2}{\partial c_2}) + \tilde{a}^3(\frac{\partial r_3}{\partial c_2})}{1 + r_1\tilde{a} + r_2\tilde{a}^2 + r_3\tilde{a}^3}]\,,\end{aligned} \tag{B.35}$$

which on substituting a by $\tilde{a}$ (B.14), demanding self-consistency principle, yields

$$\frac{\partial r_1}{\partial c_2} = 0, \quad \frac{\partial r_2}{\partial c_2} = \frac{3d}{2b}, \quad \frac{\partial r_3}{\partial c_2} = \frac{4d_1}{3b} + 3dlog\frac{\mu}{M} + 3\frac{dr_1}{2b} - r_1 - \frac{5dc}{3b}\,. \tag{B.36}$$

The same technique will apply for deriving partial derivatives of $\mathcal{M}$ with respect to c_3 and from there we will get the following results

$$\frac{\partial r_1}{\partial c_3} = 0, \quad \frac{\partial r_2}{\partial c_3} = 0, \quad \frac{\partial r_3}{\partial c_3} = \frac{5d}{6b}. \tag{B.37}$$

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